MP-282
Dynamic Modeling and Control of Multirotor Aerial Vehicles
Chapter 4: Rotational Motion

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São José dos Campos - SP
2020

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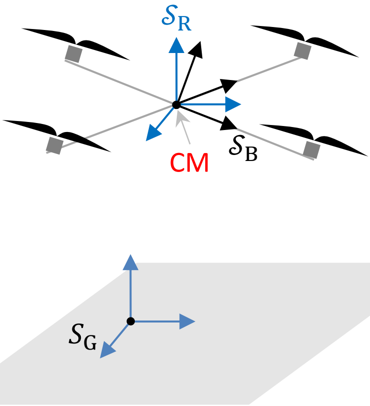
Coordinate Systems

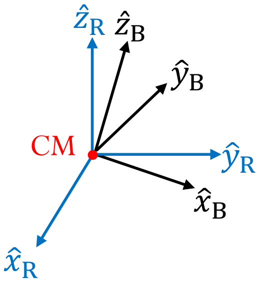
In this section, we consider the body CCS $\mathcal{S}_{\mathrm{B}} \triangleq \{B; \hat{x}_{\mathrm{B}}, \hat{y}_{\mathrm{B}}, \hat{z}_{\mathrm{B}}\}$ and the reference CCS $\mathcal{S}_{\mathrm{R}} \triangleq \{R; \hat{x}_{\mathrm{R}}, \hat{y}_{\mathrm{R}}, \hat{z}_{\mathrm{R}}\}$, where $R \equiv B \equiv \mathrm{CM}$ and $\mathcal{S}_{\mathrm{R}}$ is parallel to $\mathcal{S}_{\mathrm{G}}$.



Attitude Representations . . .

Attitude Representations

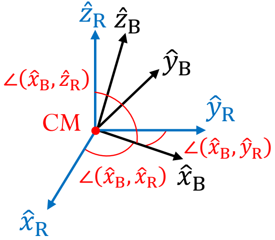
How to represent the attitude of $\mathcal{S}_{\mathrm{B}}$ with respect to (w.r.t.) $\mathcal{S}_{\mathrm{R}}$?



Attitude Matrix

By writting the versors of $\mathcal{S}_\mathrm{B}$ in terms of the versors of $\mathcal{S}_\mathrm{R}$, we obtain:

$$\begin{aligned}
\hat{x}_\mathrm{B} &= C_{xx}\hat{x}_R + C_\mathrm{xy}\hat{y}_R + C_{xz}\hat{z}_R \\
\hat{y}_\mathrm{B} &= C_{yx}\hat{x}_R + C_\mathrm{yy}\hat{y}_R + C_{yz}\hat{z}_R \\
\hat{z}_\mathrm{B} &= C_{zx}\hat{x}_R + C_\mathrm{zy}\hat{y}_R + C_{zz}\hat{z}_R
\end{aligned}$$



where $C_{ij} \triangleq \cos\angle(\hat{i}_\mathrm{B}, \hat{j}_\mathrm{R})$, for $i, j \in \{x, y, z\}$, are direction cosines.

Conclusion: The direction cosines C_{ij} completely describe the attitude of $\mathcal{S}_\mathrm{B}$ with respect to $\mathcal{S}_\mathrm{R}$.

Therefore, the attitude of $\mathcal{S}_{\mathrm{B}}$ with respect to $\mathcal{S}_{\mathrm{R}}$ can be represented by the so-called attitude matrix:

$$\mathbf{D}^{\mathrm{B/R}} \triangleq \begin{bmatrix} C_{xx} & C_{xy} & C_{xz} \\ C_{yx} & C_{yy} & C_{yz} \\ C_{zx} & C_{zy} & C_{zz} \end{bmatrix}$$

which in the literature is also called:

- Direction Cosine Matrix (DCM)
- Rotation Matrix

Attitude Representations

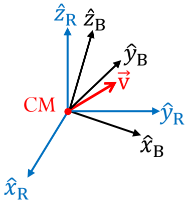
Attitude Matrix

Properties:

1. Transformation of representations

Consider an arbitrary geometric vector $\vec{v}$. Its algebraic representations $\mathbf{v}_{\mathrm{B}}$ and $\mathbf{v}_{\mathrm{R}}$ are related by

$$\mathbf{v}_{\mathrm{B}} = \mathbf{D}^{\mathrm{B/R}}\mathbf{v}_{\mathrm{R}} \quad \text{or} \quad \mathbf{v}_{\mathrm{R}} = \left(\mathbf{D}^{\mathrm{B/R}}\right)^{-1}\mathbf{v}_{\mathrm{B}}$$

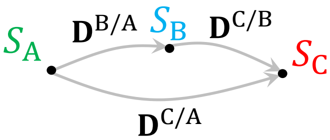


For the sake of consistency, $\left(\mathbf{D}^{\mathrm{B/R}}\right)^{-1} \equiv \mathbf{D}^{\mathrm{R/B}}$.

2. Successive Rotations

Consider the following rotations and the respective representations:

- from $\mathcal{S}_\mathrm{A}$ to $\mathcal{S}_\mathrm{C}$: $\mathbf{D}^{\mathrm{C}/\mathrm{A}}$
- from $\mathcal{S}_\mathrm{A}$ to $\mathcal{S}_\mathrm{B}$: $\mathbf{D}^{\mathrm{B}/\mathrm{A}}$
- from $\mathcal{S}_\mathrm{B}$ to $\mathcal{S}_\mathrm{C}$: $\mathbf{D}^{\mathrm{C}/\mathrm{B}}$



We can show that (see reference [1]):

$$\mathbf{D}^{\mathrm{C}/\mathrm{A}} = \mathbf{D}^{\mathrm{C}/\mathrm{B}}\mathbf{D}^{\mathrm{B}/\mathrm{A}}$$

3. Orthonormality

The attitude matrix $\mathbf{D}^{\mathrm{B/R}}$ is said to be orthonormal because its columns (or rows) are orthogonal to one another and have unit norm, *i.e.*,

$$\left(\mathbf{D}^{\mathrm{B/R}}\right)^{\mathrm{T}} \mathbf{D}^{\mathrm{B/R}} = \mathbf{I}_3$$

It implies in:

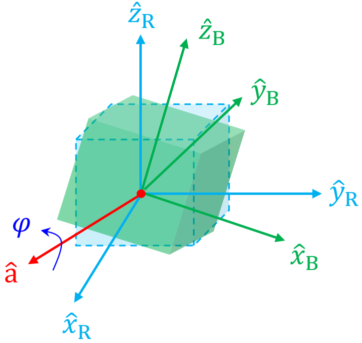
- $\left(\mathbf{D}^{\mathrm{B/R}}\right)^{-1} = \left(\mathbf{D}^{\mathrm{B/R}}\right)^{\mathrm{T}}$
- $\det\left(\mathbf{D}^{\mathrm{B/R}}\right) = 1$

i.e., $\mathbf{D}^{\mathrm{B/R}} \in \mathrm{SO}(3)$, where $\mathrm{SO}(3)$ denotes the Special Orthogonal Group [1].

Principal (Euler) Axis-Angle

Euler Theorem:

The general angular displacement of a rigid body with a fixed point can be described by a single rotation angle φ around an axis $\hat{a}$ passing through that point.



Therefore, the attitude of $\mathcal{S}_\mathrm{B}$ w.r.t. $\mathcal{S}_\mathrm{R}$ can be represented by the pair:

$$(\varphi, \mathbf{a})$$

where $\varphi \in \mathbb{R}$ is the principal Euler angle and $\mathbf{a}$ – which is the representation of $\hat{a}$ in any of the two aforementioned CCSs[1] – is the principal Euler axis.

[1] Note that since the rotation of $\mathcal{S}_\mathrm{B}$ with respect to $\mathcal{S}_\mathrm{R}$ is strictly around $\hat{a}$, the representations $\mathbf{a}_\mathrm{B}$ and $\mathbf{a}_\mathrm{R}$ are the same. Try to do it with a coordinate axis!

Relation with the Attitude Matrix:

Given $(\varphi, \mathbf{a})$, we obtain $\mathbf{D}^{\mathrm{B/R}}$ by (see reference [1])

$$\mathbf{D}^{\mathrm{B/R}} = \cos\varphi \mathbf{I}_3 + (1 - \cos\varphi)\mathbf{a}\mathbf{a}^{\mathrm{T}} - \sin\varphi[\mathbf{a}\times]$$

where

$$[\mathbf{a}\times] \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

On the other hand, given $\mathbf{D}^{\mathrm{B/R}}$, we can obtain $(\varphi, \mathbf{a})$ by

$$\varphi = \mathrm{acos}\frac{\mathrm{tr}\ \mathbf{D}^{\mathrm{B/R}} - 1}{2}$$

and

- if $\mathrm{tr}\ \mathbf{D}^{\mathrm{B/R}} = 3$ then $\mathbf{a}$ is indefinite
- if $\mathrm{tr}\ \mathbf{D}^{\mathrm{B/R}} = -1$ then

$$a_1 = \pm\sqrt{\frac{1 + D_{11}}{2}} \qquad a_2 = \pm\sqrt{\frac{1 + D_{22}}{2}} \qquad a_3 = \pm\sqrt{\frac{1 + D_{33}}{2}}$$

$$a_1 a_2 = \frac{D_{12}}{2} \qquad a_2 a_3 = \frac{D_{23}}{2} \qquad a_3 a_1 = \frac{D_{31}}{2}$$

- if $\mathrm{tr}\ \mathbf{D}^{\mathrm{B/R}} \neq -1$ and $\neq 3$ then

$$a_1 = \frac{D_{23} - D_{32}}{2\sin\varphi} \qquad a_2 = \frac{D_{31} - D_{13}}{2\sin\varphi} \quad a_3 = \frac{D_{12} - D_{21}}{2\sin\varphi}$$

Remark:

There exists an ambiguity in the axis-angle representation:

$(\varphi, \mathbf{a})$ and $(-\varphi, -\mathbf{a})$ represent the same attitude.

Euler Parameters (Quaternion)

Definition:

$$\mathbf{q} = \begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix} \in \mathbb{R}^4$$

where $\boldsymbol{\varepsilon} \in \mathbb{R}^3$ and $\eta \in \mathbb{R}$ are

$$\boldsymbol{\varepsilon} \triangleq \mathbf{a} \sin \frac{\varphi}{2}$$

$$\eta \triangleq \cos \frac{\varphi}{2}$$

Properties:

1. Relation with the Attitude Matrix:

Given $\mathbf{q}$, we obtain $\mathbf{D}^{\mathrm{B/R}}$ by

$$\mathbf{D}^{\mathrm{B/R}} = \left(\eta^2 - \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\varepsilon}\right)\mathbf{I}_3 + 2\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{\mathrm{T}} - 2\eta\left[\boldsymbol{\varepsilon}\times\right]$$

On the other hand, given $\mathbf{D}^{\mathrm{B/R}}$, we can obtain $\mathbf{q}$ by

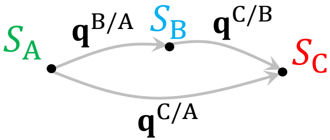
$$\eta = \pm\frac{1}{2}\sqrt{1 + \mathrm{tr}\ \mathbf{D}^{\mathrm{B/R}}}$$

$$\boldsymbol{\varepsilon} = \frac{1}{4\eta}\begin{bmatrix} D_{23} - D_{32} \\ D_{31} - D_{13} \\ D_{12} - D_{21} \end{bmatrix}$$

2. Successive Rotations

Consider the following rotations and the respective representations:

- from $\mathcal{S}_\mathrm{A}$ to $\mathcal{S}_\mathrm{C}$: $\mathbf{q}^{\mathrm{C/A}}$
- from $\mathcal{S}_\mathrm{A}$ to $\mathcal{S}_\mathrm{B}$: $\mathbf{q}^{\mathrm{B/A}} \triangleq (\varepsilon_1, \eta_1)$
- from $\mathcal{S}_\mathrm{B}$ to $\mathcal{S}_\mathrm{C}$: $\mathbf{q}^{\mathrm{C/B}} \triangleq (\varepsilon_2, \eta_2)$



We can show that:

$$\mathbf{q}^{\mathrm{C/A}} = \begin{bmatrix} \eta_2\varepsilon_1 + \eta_1\varepsilon_2 + [\varepsilon_1 \times]\,\varepsilon_2 \\ \eta_1\eta_2 - \varepsilon_1^{\mathrm{T}}\varepsilon_2 \end{bmatrix}$$

3. Unit Norm

The attitude quaternion $\mathbf{q}$ has unit norm:

$$\mathbf{q}^{\mathrm{T}}\mathbf{q} = 1$$

i.e., $\mathbf{q} \in \mathbb{S}^3 \subset \mathbb{R}^4$ [2].

[2] The symbol $\mathbb{S}^3$ denotes the 3-Sphere.

Remark:

There is an ambiguity in the quaternion representation:

$\mathbf{q}$ and $-\mathbf{q}$ represent the same attitude.

Gibbs Vector

Definition:

$$\mathbf{g} \triangleq \mathbf{a} \tan \frac{\varphi}{2}$$

In the literature, the vector $\mathbf{g} \in \mathbb{R}^3$ is also known as

- Rodrigues Parameters
- Euler-Rodrigues Parameters

Attitude Representations

Gibbs Vector

Properties:

1. Relation with the Attitude Matrix:

Given $\mathbf{g}$, we obtain $\mathbf{D}^{\mathrm{B/R}}$ by

$$\mathbf{D}^{\mathrm{B/R}} = \frac{\left(1 - \mathbf{g}^{\mathrm{T}}\mathbf{g}\right)\mathbf{I}_3 + 2\mathbf{g}\mathbf{g}^{\mathrm{T}} - 2\left[\mathbf{g}\times\right]}{1 + \mathbf{g}^{\mathrm{T}}\mathbf{g}}$$

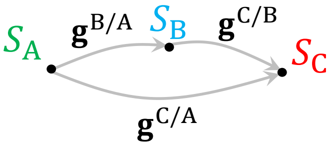
On the other hand, given $\mathbf{D}^{\mathrm{B/R}}$, we obtain $\mathbf{g}$ by

$$\mathbf{g} = \frac{1}{1 + \mathrm{tr}\ \mathbf{D}^{\mathrm{B/R}}}\begin{bmatrix} D_{23} - D_{32} \\ D_{31} - D_{13} \\ D_{12} - D_{21} \end{bmatrix}$$

2. Successive Rotations

Consider the following rotations and the respective representations:

- from $\mathcal{S}_\mathrm{A}$ to $\mathcal{S}_\mathrm{C}$: $\mathbf{g}^{\mathrm{C/A}}$
- from $\mathcal{S}_\mathrm{A}$ to $\mathcal{S}_\mathrm{B}$: $\mathbf{g}^{\mathrm{B/A}} \triangleq \mathbf{g}_1$
- from $\mathcal{S}_\mathrm{B}$ to $\mathcal{S}_\mathrm{C}$: $\mathbf{g}^{\mathrm{C/B}} \triangleq \mathbf{g}_2$



We can show that:

$$\mathbf{g}^{\mathrm{C/A}} = \frac{\mathbf{g}_1 + \mathbf{g}_2 - [\mathbf{g}_2 \times]\,\mathbf{g}_1}{1 - \mathbf{g}_1^{\mathrm{T}}\mathbf{g}_2}$$

Remarks:

- There is no ambiguity in $\mathbf{g}$.
- There exists a singularity in $\mathbf{g}$ at $\varphi = \pm 180^o(2i+1)$, $\forall i$.

Euler Angles

Elemetary Rotations:

They are rotations around the coordinate axes. Denote by $\mathbf{D}_i(\varrho)$ the elementary rotation matrix representing the displacement of an angle ϱ around axis i, for $i \in \{1, 2, 3\}$.

We can show that:

$$\mathbf{D}_1(\varrho) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\varrho & \mathrm{s}\varrho \\ 0 & -\mathrm{s}\varrho & \mathrm{c}\varrho \end{bmatrix} \quad \mathbf{D}_2(\varrho) = \begin{bmatrix} \mathrm{c}\varrho & 0 & -\mathrm{s}\varrho \\ 0 & 1 & 0 \\ \mathrm{s}\varrho & 0 & \mathrm{c}\varrho \end{bmatrix}$$

$$\mathbf{D}_3(\varrho) = \begin{bmatrix} \mathrm{c}\varrho & \mathrm{s}\varrho & 0 \\ -\mathrm{s}\varrho & \mathrm{c}\varrho & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Three-Dimensional Attitude Representation:

Three-dimensional attitude can be represented by a sequence of three elementary rotation about three consecutively different axes. There are 12 possible sequences:

$$313, 212, 121, 131, 323, 232$$

$$123, 321, 132, 312, 231, 213$$

We will adopt the sequence 123.

Relation with the Attitude Matrix:

Given the Euler angles (123) (ϕ, θ, ψ), we obtain $\mathbf{D}^{\mathrm{B/R}}$ by

$$\begin{aligned}\mathbf{D}^{\mathrm{B/R}} &= \mathbf{D}_3(\psi)\mathbf{D}_2(\theta)\mathbf{D}_1(\phi) \\ &= \begin{bmatrix} \mathrm{c}\psi\mathrm{c}\theta & \mathrm{c}\psi\mathrm{s}\theta\mathrm{s}\phi + \mathrm{s}\psi\mathrm{c}\phi & -\mathrm{c}\psi\mathrm{s}\theta\mathrm{c}\phi + \mathrm{s}\psi\mathrm{s}\phi \\ -\mathrm{s}\psi\mathrm{c}\theta & -\mathrm{s}\psi\mathrm{s}\theta\mathrm{s}\phi + \mathrm{c}\psi\mathrm{c}\phi & \mathrm{s}\psi\mathrm{s}\theta\mathrm{c}\phi + \mathrm{c}\psi\mathrm{s}\phi \\ \mathrm{s}\theta & -\mathrm{c}\theta\mathrm{s}\phi & \mathrm{c}\theta\mathrm{c}\phi \end{bmatrix}\end{aligned}$$

On the other hand, given $\mathbf{D}^{\mathrm{B/R}} = [D_{ij}]$, we have

$$\phi = -\mathrm{atan}\,\frac{D_{32}}{D_{33}},\quad 0^o \leq \phi < 360^o$$

$$\theta = \mathrm{asin}\,D_{31},\quad -90^o < \theta < 90^o$$

$$\psi = -\mathrm{atan}\,\frac{D_{21}}{D_{11}},\quad 0^o \leq \psi < 360^o$$

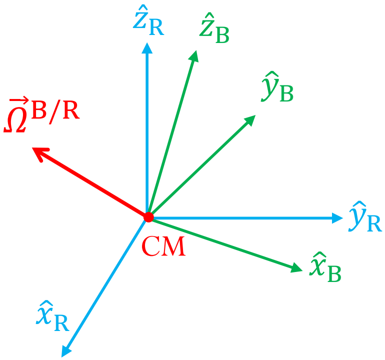
Remarks:

- It has no ambiguity.
- For visualization, Euler angles are the best attitude representation, since the alternatives have no obvious physical meaning.
- For simulation, Euler angles are the worst attitude representation, since it has singularity at $\theta = 90^o$ (we are going to see it in the next section) and its kinematics equation is the most nonlinear one.

Attitude Kinematics ...

Definition

- Let $\overrightarrow{\mathbf{\Omega}}^{\mathrm{B/R}}$ denote the angular velocity of $\mathcal{S}_{\mathrm{B}}$ with respect to $\mathcal{S}_{\mathrm{R}}$.
- The attitude Kinematics is the motion of $\mathcal{S}_{\mathrm{B}}$ w.r.t. $\mathcal{S}_{\mathrm{R}}$ as a function of $\overrightarrow{\mathbf{\Omega}}^{\mathrm{B/R}}$.



Kinematics in Attitude Matrix

One can show that (see reference [2])

$$\dot{\mathbf{D}}^{\mathrm{B/R}} = -\left[\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}}\times\right]\mathbf{D}^{\mathrm{B/R}}$$

Kinematics in Axis-Angle

One can show that (see reference [1], p. 24–25)

$$\dot{\varphi} = \mathbf{a}^{\mathrm{T}} \boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}}$$

$$\dot{\mathbf{a}} = \frac{1}{2}\left([\mathbf{a}\times] - \cot\frac{\varphi}{2} [\mathbf{a}\times][\mathbf{a}\times] \right) \boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}}$$

Kinematics in Quaternion

One can show that

$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{W}\mathbf{q}$$

where

$$\mathbf{W} \triangleq \begin{bmatrix} -\left[\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}}\times\right] & \boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}} \\ -\left(\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}}\right)^{\mathrm{T}} & 0 \end{bmatrix}$$

Kinematics in Gibbs Vector

One can show that

$$\dot{\mathbf{g}} = \frac{1}{2}\left(\mathbf{g}\mathbf{g}^{\mathrm{T}} + [\mathbf{g}\times] + \mathbf{I}_3\right)\mathbf{\Omega}_{\mathrm{B}}^{\mathrm{B/R}}$$

Kinematics in Euler Angles 123

One can show that

$$\dot{\boldsymbol{\alpha}} = \mathbf{A}\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}}$$

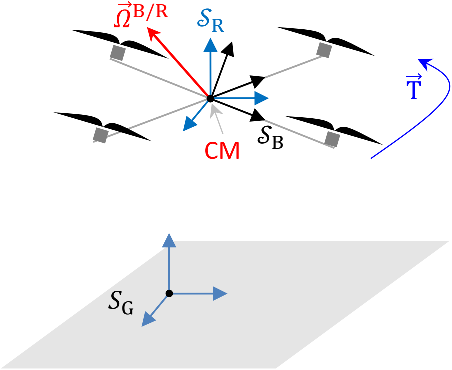
where $\boldsymbol{\alpha} \triangleq [\phi \ \theta \ \psi]^{\mathrm{T}}$ and

$$\mathbf{A} \triangleq \begin{bmatrix} \mathrm{c}\psi/\mathrm{c}\theta & -\mathrm{s}\psi/\mathrm{c}\theta & 0 \\ \mathrm{s}\psi & \mathrm{c}\psi & 0 \\ -\mathrm{c}\psi\mathrm{s}\theta/\mathrm{c}\theta & \mathrm{s}\psi\mathrm{s}\theta/\mathrm{c}\theta & 1 \end{bmatrix}$$

Attitude Dynamics . . .

Definition

Let $\overrightarrow{T}$ denote the total torque acting on the MAV. The attitude dynamics describe the time variation of $\overrightarrow{\Omega}^{\mathrm{B/R}}$ as a function of $\overrightarrow{T}$.



A Simple Model

By assuming that

- The MAV airframe is rigid and has inertia matrix $\mathbf{J}_{\mathrm{B}}$ (in $\mathcal{S}_{\mathrm{B}}$).
- The angular momenta of its rotors are negligible.
- $\overrightarrow{T} = \overrightarrow{T}^c + \overrightarrow{T}^d$, where $\overrightarrow{T}^c$ is the control torque and $\overrightarrow{T}^d$ is the disturbance torque.

the MAV's (attitude) dynamic equation is given in $\mathcal{S}_{\mathrm{B}}$ by

$$\dot{\mathbf{\Omega}}_{\mathrm{B}}^{\mathrm{B/R}} = \mathbf{J}_{\mathrm{B}}^{-1} \left[\left(\mathbf{J}_{\mathrm{B}} \mathbf{\Omega}_{\mathrm{B}}^{\mathrm{B/R}} \right) \times \right] \mathbf{\Omega}_{\mathrm{B}}^{\mathrm{B/R}} + \mathbf{J}_{\mathrm{B}}^{-1} \left(\mathbf{T}_{\mathrm{B}}^{\mathrm{c}} + \mathbf{T}_{\mathrm{B}}^{\mathrm{d}} \right)$$

References. . .

References

[1] Hughes, P. C. **Spacecraft Attitude Dynamics**. Dover, 2004.

[2] Shuster, M. Survey of Attitude Representations. **The Journal of Astronautical Sciences**, Vol. 41, No. 4, 1993.

Thanks!