MP-282

Dynamic Modeling and Control of Multirotor Aerial Vehicles

Chapter 5: Translational Motion, Euler Lagrange, and Summary

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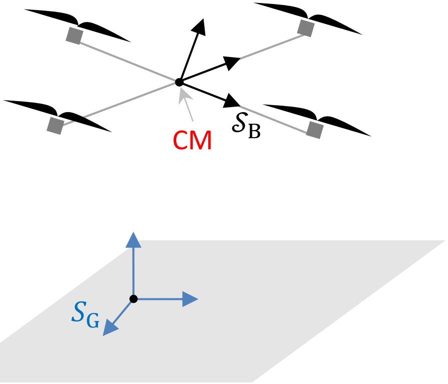
Coordinate Systems . . .

Coordinate Systems

Two Cartesian Coordinate Systems

In this section, we consider the body CCS $\mathcal{S}_{\mathrm{B}} \triangleq \{B; \hat{x}_{\mathrm{B}}, \hat{y}_{\mathrm{B}}, \hat{z}_{\mathrm{B}}\}$ and the ground CCS $\mathcal{S}_{\mathrm{G}} \triangleq \{G; \hat{x}_{\mathrm{G}}, \hat{y}_{\mathrm{G}}, \hat{z}_{\mathrm{G}}\}$, where $B \equiv \mathrm{CM}$.

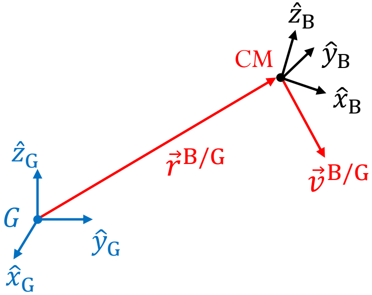


Kinematic Equation . . .

Kinematic Equation

Scope

- $\overrightarrow{r}^{\mathrm{B/G}}$: position vector of $\mathcal{S}_{\mathrm{B}}$ w.r.t. $\mathcal{S}_{\mathrm{G}}$.
- $\overrightarrow{v}^{\mathrm{B/G}} \triangleq \frac{{}^{\mathrm{G}}d}{dt}\overrightarrow{r}^{\mathrm{B/G}}$: velocity vector of $\mathcal{S}_{\mathrm{B}}$ w.r.t. $\mathcal{S}_{\mathrm{G}}$ as observed from $\mathcal{S}_{\mathrm{G}}$.



The kinematics are represented by the time evolution of $\overrightarrow{r}^{\mathrm{B/G}}$ as a function of $\overrightarrow{v}^{\mathrm{B/G}}$.

Kinematic Equation

Model

The translational kinematics of $\mathcal{S}_{\mathrm{B}}$ w.r.t. $\mathcal{S}_{\mathrm{G}}$ are modeled in $\mathcal{S}_{\mathrm{G}}$ by

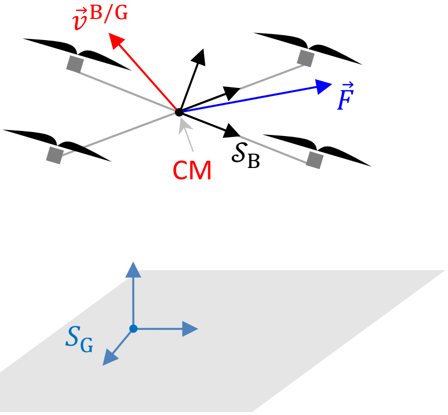
$$\dot{\mathbf{r}}_{\mathrm{G}}^{\mathrm{B/G}} = \mathbf{v}_{\mathrm{G}}^{\mathrm{B/G}}$$

Dynamic Equation . . .

Dynamic Equation

Scope

Denote the resulting external force by $\overrightarrow{F}$. The translational dynamics are represented by the time evolution of $\overrightarrow{v}^{\mathrm{B/G}}$ as a function of $\overrightarrow{F}$.



Dynamic Equation

Assumptions

Assume that

1. $\mathcal{S}_{\mathrm{G}}$ is an inertial frame.
2. $\overrightarrow{F} = \overrightarrow{F}^c + \overrightarrow{F}^g + \overrightarrow{F}^d$ [1].
3. $\overrightarrow{F}^c$, $\overrightarrow{F}^g$, $\overrightarrow{F}^d$ are modeled as in Chapter 2.

Newton's Law

The Second Newton's Law gives

$$\frac{{}^{\mathrm{G}}d}{dt}\overrightarrow{v}^{\mathrm{B/G}} = \frac{1}{m}\overrightarrow{F}$$

[1] For simplicity, we are not considering specific forces, *e.g.*, due to a balloon, a tether, or blading flapping. However, the extension should be smooth.

Dynamic Equation

Model

The translational dynamics of $\mathcal{S}_{\mathrm{B}}$ w.r.t. $\mathcal{S}_{\mathrm{G}}$ are modeled in $\mathcal{S}_{\mathrm{G}}$ by

$$\dot{\mathbf{v}}_{\mathrm{G}}^{\mathrm{B/G}} = \frac{1}{m}\left(\mathbf{D}^{\mathrm{B/G}}\right)^{\mathrm{T}}\mathbf{F}_{\mathrm{B}}^{c} + \frac{1}{m}\mathbf{F}_{\mathrm{G}}^{g} + \frac{1}{m}\mathbf{F}_{\mathrm{G}}^{d}$$

where $\mathbf{D}^{\mathrm{B/G}} \equiv \mathbf{D}^{\mathrm{B/R}}$, which is known from Chapter 4, and the models for $\mathbf{F}_{\mathrm{B}}^{c}$, $\mathbf{F}_{\mathrm{G}}^{g}$, as well as $\mathbf{F}_{\mathrm{G}}^{d}$ are given in Chapter 2.

Euler-Lagrange Formulation . . .

Euler-Lagrange Formulation[2]

Generalized Coordinates

Define the vector of generalized coordinates:

$$\mathbf{x} \triangleq \left[\left(\mathbf{r}_{\mathrm{G}}^{\mathrm{B/G}} \right)^{\mathrm{T}} \ \boldsymbol{\alpha}^{\mathrm{T}} \right]^{\mathrm{T}}$$

where $\boldsymbol{\alpha} \triangleq [\phi \ \theta \ \psi]^{\mathrm{T}}$ is the vector of Euler angles 123.

[2] This section is based on reference [1], pp. 25–28.

Euler-Lagrange Formulation

Lagrangean

The Lagrangean of our six-DOF system is given by:

$$\mathcal{L} \triangleq K_t + K_r - P$$

where

- $K_t \triangleq \frac{m}{2}\left(\dot{\mathbf{r}}_{\mathrm{G}}^{\mathrm{B/G}}\right)^{\mathrm{T}} \dot{\mathbf{r}}_{\mathrm{G}}^{\mathrm{B/G}}$ is the translational kinetic energy.
- $K_r \triangleq \frac{1}{2}\left(\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}}\right)^{\mathrm{T}} \mathbf{J}_{\mathrm{B}}\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}}$ is the rotational kinetic energy.
- $P \triangleq mg\mathbf{e}_3^{\mathrm{T}}\mathbf{r}_{\mathrm{G}}^{\mathrm{B/G}}$ is the potential energy.

Euler-Lagrange Formulation

Attitude Kinematic (Euler angles 123)

From Chapter 4, we know that [3]

$$\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}} = \mathcal{A}(\boldsymbol{\alpha})^{-1}\dot{\boldsymbol{\alpha}}$$

where

$$\mathcal{A}(\boldsymbol{\alpha})^{-1} = \begin{bmatrix} \mathrm{c}\theta\mathrm{c}\psi & \mathrm{s}\psi & 0 \\ -\mathrm{c}\theta\mathrm{s}\psi & \mathrm{c}\psi & 0 \\ \mathrm{s}\theta & 0 & 1 \end{bmatrix}$$

[3] A similar derivation can be done for any attitude representation seen in Chapter 4.

Euler-Lagrange Formulation

Rotational Kinetic Energy

Using the above equation, it can be rewritten in the form

$$K_r = \frac{1}{2}\dot{\boldsymbol{\alpha}}^{\mathrm{T}}\mathbb{J}\dot{\boldsymbol{\alpha}}$$

where

$$\mathbb{J} \triangleq \mathcal{A}(\boldsymbol{\alpha})^{-\mathrm{T}}\mathbf{J}_{\mathrm{B}}\mathcal{A}(\boldsymbol{\alpha})^{-1}$$

Euler-Lagrange Formulation

Euler-Lagrange Equation

The Euler-Lagrange equation for the system under consideration is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \begin{bmatrix} \mathbf{F}_{\mathrm{G}} \\ \mathbf{T}_{\mathrm{B}} \end{bmatrix}$$

where $\mathbf{F}_{\mathrm{G}}$ is the $\mathcal{S}_{\mathrm{G}}$ representation of the resulting force and $\mathbf{T}_{\mathrm{B}}$ is the $\mathcal{S}_{\mathrm{B}}$ representation of the resulting torque.

Remark:

Since $\mathcal{L}$ does not contain terms involving both $\dot{\mathbf{r}}_{\mathrm{G}}^{\mathrm{B/G}}$ and $\dot{\alpha}$ together, we can separate the above equation into two parts: one for the translation and the other one for the rotation.

Euler-Lagrange Formulation

Translation

Define the translation Lagrangean as $\mathcal{L}_t \triangleq K_t - P$. Therefore, according to the Euler-Lagrange formulation, the translational equation of motion is obtained from

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}_t}{\partial \dot{\mathbf{r}}_{\mathrm{G}}^{\mathrm{B/G}}}\right) - \frac{\partial \mathcal{L}_t}{\partial \mathbf{r}_{\mathrm{G}}^{\mathrm{B/G}}} = \mathbf{F}_{\mathrm{G}}$$

as

$$m\ddot{\mathbf{r}}_{\mathrm{G}}^{\mathrm{B/G}} = \mathbf{F}_{\mathrm{G}} - mg\mathbf{e}_3$$

which coincides with the Newton's Second Law.

Euler-Lagrange Formulation

Rotation

Define the rotation Lagrangean as $\mathcal{L}_r \triangleq K_r$. According to the Euler-Lagrange formulation, the rotational equation of motion is obtained from

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}_r}{\partial \dot{\boldsymbol{\alpha}}}\right) - \frac{\partial \mathcal{L}_r}{\partial \boldsymbol{\alpha}} = \mathbf{T}_{\mathrm{B}}$$

as

$$\mathbb{J}\ddot{\boldsymbol{\alpha}} + C\left(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}\right)\dot{\boldsymbol{\alpha}} = \mathbf{T}_{\mathrm{B}}$$

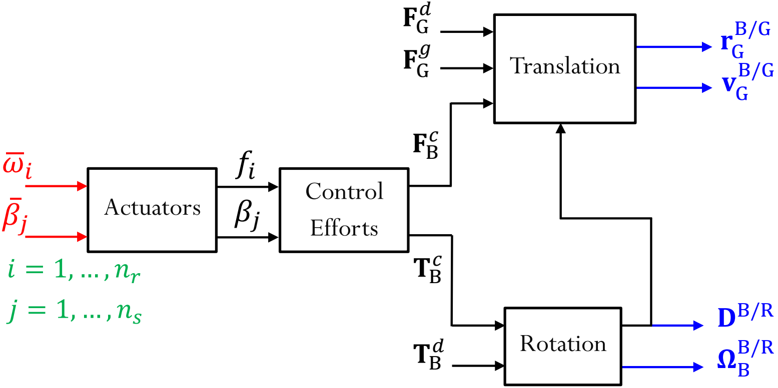
where

$$C\left(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}\right) \triangleq -\frac{1}{2}\dot{\boldsymbol{\alpha}}^{\mathrm{T}}\left(\mathbf{A}(\boldsymbol{\alpha})^{-\mathrm{T}}\mathbf{J}_{\mathrm{B}}\frac{\partial}{\partial \boldsymbol{\alpha}}\left(\mathbf{A}(\boldsymbol{\alpha})^{-1}\right) + \frac{\partial}{\partial \boldsymbol{\alpha}}\left(\mathbf{A}(\boldsymbol{\alpha})^{-T}\right)\mathbf{J}_{\mathrm{B}}\mathbf{A}(\boldsymbol{\alpha})\right)$$

Summary . . .

Summary

Overall MAV Modeling [4]



[4]This block diagram does not consider a balloon neither a tether.

References . . .

References

[1] Goldstein, H. **Classical Mechanics**. Addison-Wesley, 1950.

[2] Carrillo, L.R.G., Lopez, A.E.D., Lozano, R., Pegard, C. **A Quad Rotorcraft Control - Vision-Based Hovering and Navigation**. Springer, 2013.

Thanks!