MP-282
Dynamic Modeling and Control of Multirotor Aerial Vehicles
Chapter 6: Flight Control

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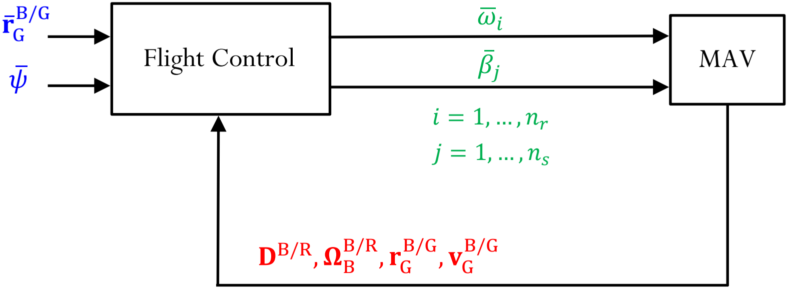
Problem Statement

The problem is to design feedback control laws for $\bar{\omega}_i$, $i = 1, ..., n_r$, and $\bar{\beta}_j$, $j = 1, ..., n_s$, for the MAV to appropriately track the desired position command $\bar{\mathbf{r}}_{\mathrm{G}}^{\mathrm{B/G}}$ and heading command $\bar{\psi}$.



Problem Definition

Assumptions

1. The MAV dynamic and kinematic models are known[1].
2. The state variables $\mathbf{D}^{B/R}, \mathbf{\Omega}_{B}^{B/R}, \mathbf{r}_{G}^{B/G}, \mathbf{v}_{G}^{B/G}$ are available for feedback[2].
3. The MAV has cascaded dynamics such that its rotation affects its translation[3].
4. Time-Scale Separation (TSS): the closed-loop rotational dynamics are much faster than the closed-loop translational dynamics[4].

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[1] See Chapters 2–4 and note that by this assumption, the disturbances are zero and the plant parameters are perfectly known.

[2] In principle, we can use some kind of observer, together with sensor measurements, to estimate these variables.

[3] By this assumption, the present chapter covers the fixed-rotor MAVs.

[4] This can be enforced by appropriately tuning the controllers.

Problem Definition

Design Requirements

1. The closed-loop system must be stable.
2. The closed-loop system must satisfy performance requirements, *e.g.*, in terms of overshoot, peak instant, and steady-state error.
3. The closed-loop system must assure some performance level and stability even in the presence of disturbances and uncertainties [5].
4. The control law(s) must be implementable in real-time embedded systems.

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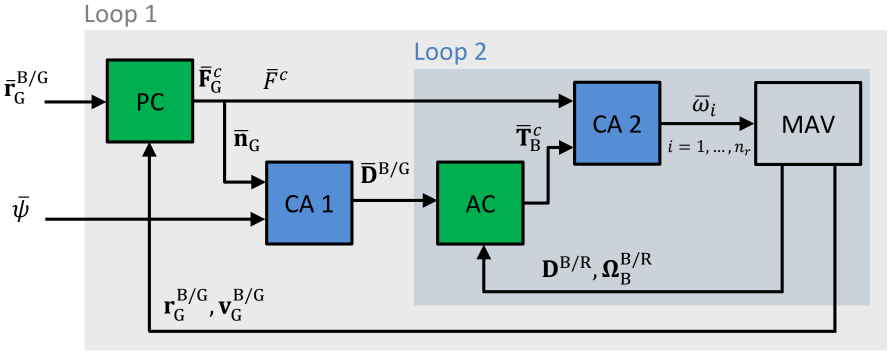
[5] The design presented in this chapter does not address robustness explicitly, but feedback provides this characteristic anyway. We postpone an explicit robust design to Chapter 8.

Problem Solution . . .

Control Architecture

Considering Assumptions 2–4 and focusing on fixed-rotor MAVs, we adopt the control architecture:



Legend: PC - position controller.
AC - attitude controller.
CA 1, CA 2 - control allocators.

Attitude Control Law

The rotational dynamics were modeled in Chapter 4 by:

$$\dot{\mathbf{D}}^{\mathrm{B/R}} = -\left[\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}} \times\right] \mathbf{D}^{\mathrm{B/R}}$$

$$\dot{\boldsymbol{\Omega}}_{\mathrm{B}}^{\mathrm{B/R}} = \mathbf{J}_{\mathrm{B}}^{-1} \left[\left(\mathbf{J}_{\mathrm{B}} \boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}}\right) \times\right] \boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}} + \mathbf{J}_{\mathrm{B}}^{-1} \left(\mathbf{T}_{\mathrm{B}}^{\mathrm{c}} + \mathbf{T}_{\mathrm{B}}^{\mathrm{d}}\right)$$

Assume here that:

5. $\bar{\mathbf{D}}^{\mathrm{B/G}}$ is constant (TSS; it is useful for the stability study).
6. $\omega_i = \bar{\omega}_i,\ i = 1, ..., n_r$ [6].
7. $\mathbf{T}_{\mathrm{G}}^{d} = \mathbf{0}$.

[6] Note that this stems from a time-scale separation assumption as well.

Problem Solution

Considering that the dimensional parameters are exactly known, Assumption 6 implies in

$$\mathbf{T}_{\mathrm{B}}^{c} = \bar{\mathbf{T}}_{\mathrm{B}}^{c}$$

From this implication and Assumption 7, we obtain the *design model*[7]:

$$\dot{\mathbf{D}}^{\mathrm{B/R}} = -\left[\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}} \times\right] \mathbf{D}^{\mathrm{B/R}} \tag{1}$$

$$\dot{\boldsymbol{\Omega}}_{\mathrm{B}}^{\mathrm{B/R}} = \mathbf{J}_{\mathrm{B}}^{-1} \left[\left(\mathbf{J}_{\mathrm{B}} \boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}}\right) \times\right] \boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}} + \mathbf{J}_{\mathrm{B}}^{-1} \bar{\mathbf{T}}_{\mathrm{B}}^{\mathrm{c}} \tag{2}$$

[7] This is how we refer to the simplified model used to design the attitude control law. In our simulations, we can distinguish between it and a simulation (or ground-truth) model.

Problem Solution

Attitude Control Law

Based on the design model (2), we adopt a saturated PD control law with a feedback term for cancelling its nonlinearity when the saturation is not activated:

$$\gamma^a = \mathbf{J}_{\mathrm{B}}\mathbf{K}_1\mathbf{p} - \mathbf{J}_{\mathrm{B}}\mathbf{K}_2\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}} - \left[\left(\mathbf{J}_{\mathrm{B}}\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}}\right)\times\right]\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}} \tag{3}$$

$$\bar{\mathbf{T}}_{\mathrm{B}}^c = \mathrm{sat}_{\mathcal{T}}\left(\gamma^a\right) \tag{4}$$

where $\mathbf{p} \in \mathbb{R}^3$ is a three-dimensional parameterization[8] of the attitude control error matrix $\tilde{\mathbf{D}} \triangleq \bar{\mathbf{D}}^{\mathrm{B/R}}\mathbf{D}^{\mathrm{R/B}}$, $\mathbf{K}_1 \in \mathbb{R}^{3\times 3}$ and $\mathbf{K}_2 \in \mathbb{R}^{3\times 3}$ are the proportional and derivative gain matrices, respectively, $\mathcal{T} \subset \mathbb{R}^3$ is a torque admissible set, and $\mathrm{sat}_{\mathcal{T}}$ is the componentwise saturation function.

[8] See Section 4.2 for some examples of three-dimensional attitude representations.

Problem Solution

The admissible set can be chosen as

$$\mathcal{T} \triangleq \left\{ \mathbf{T} \in \mathbb{R}^3 : -\mathbf{T}^{\max} \preceq \mathbf{T} \preceq \mathbf{T}^{\max} \right\}$$

To reflect about how to compute $\mathbf{T}^{\max}$, for simplicity, let us consider an MAV with fixed rotors with all rotor axes parallel to $\hat{z}_{\mathrm{B}}$. We know that if the commanded torque is null, then $\bar{f}_i = \bar{f}$, $\forall i$, where $\bar{f} \triangleq mg/n_r$ is the nominal thrust.

On the contrary, if there exists a torque command, the rotors of each opposite pair will receive thryst commands which are symmetrical w.r.t. $\bar{f}$. We need now to distinguish between the real (physical) and the virtual bounds of $\bar{f}_i$. For this end, denote:

- real bounds: $f_i \in [f^{\min}, f^{\max}]$
- virtual bounds: $\bar{f}_i \in [\zeta^{\min}, \zeta^{\max}]$

Problem Solution

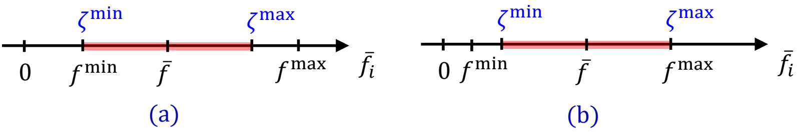
We can establish the virtual bounds as follows:

a. **if** $\bar{f} - f^{\min} < f^{\max} - \bar{f}$, **then**:

$$\zeta^{\min} = f^{\min}$$
$$\zeta^{\max} = 2\bar{f} - f^{\min}$$

b. **if** $\bar{f} - f^{\min} \geq f^{\max} - \bar{f}$, **then**:

$$\zeta^{\min} = 2\bar{f} - f^{\max}$$
$$\zeta^{\max} = f^{\max}$$



Problem Solution

Finally, to obtain the maximal torque $\mathbf{T}^{\max}$, consider the formulas for computing the resulting efforts (see Chapter 3). Then just replace f_i in those formulas by either $\zeta^{\min}$ (if the respective term is negative) or by $\zeta^{\max}$ (if the respective term is positive).

Example: For a quadcopter Q+, the maximal control force is

$$\mathbf{T}^{\max} = \begin{bmatrix} l(\zeta^{\max} - \zeta^{\min}) \\ l(\zeta^{\max} - \zeta^{\min}) \\ 2k(\zeta^{\max} - \zeta^{\min}) \end{bmatrix}$$

Remark: The above maximal bounds on the torque components are only exact if the torque command occurs about a single coordinate axis. Otherwise, if there are torque commands about more than one axis, these bounds should be reduced accordingly (why?).

Problem Solution

Regarding the stability of the closed-loop rotational dynamics:

1. By considering that the saturation of (4) is not activated and replacing (3)–(4) into (2) we obtain a linear time-invariant closed-loop rotational dynamic model. Therefore, one can see that stability, without saturation, can be reached by choosing $\mathbf{K}_1$ and $\mathbf{K}_2$ as diagonal and positive definite.

2. For a formal stability proof considering the saturation, we suggest reading section 3.5.2 of reference [1].

Problem Solution

Position Control Law

The translational dynamics have been modeled in Chapter 5 as:

$$\dot{\mathbf{r}}_{\mathrm{G}}^{\mathrm{B/G}} = \mathbf{v}_{\mathrm{G}}^{\mathrm{B/G}}$$

$$\dot{\mathbf{v}}_{\mathrm{G}}^{\mathrm{B/G}} = \frac{1}{m}\left(\mathbf{D}^{\mathrm{B/R}}\right)^{\mathrm{T}}\mathbf{F}_{\mathrm{B}}^{c} - g\mathbf{e}_3 + \frac{1}{m}\mathbf{F}_{\mathrm{G}}^{d}$$

Assume here that:

8. $\mathbf{D}^{\mathrm{B/R}} = \bar{\mathbf{D}}^{\mathrm{B/R}}$ (TSS).
9. $\omega_i = \bar{\omega}_i$, $i = 1, ..., n_r$.
10. $\mathbf{F}_{\mathrm{G}}^{d} = \mathbf{0}$.

Problem Solution

Assumptions 8–9 imply in

$$\mathbf{F}_{\mathrm{B}}^{c} = \bar{\mathbf{F}}_{\mathrm{B}}^{c}$$
$$\mathbf{F}_{\mathrm{G}}^{c} = \bar{\mathbf{F}}_{\mathrm{G}}^{c}$$

From the above implication and Assumption 10, we obtain the design model[9]:

$$\dot{\mathbf{r}}_{\mathrm{G}}^{\mathrm{B/G}} = \mathbf{v}_{\mathrm{G}}^{\mathrm{B/G}} \tag{5}$$
$$\dot{\mathbf{v}}_{\mathrm{G}}^{\mathrm{B/G}} = \frac{1}{m}\bar{\mathbf{F}}_{\mathrm{G}}^{c} - g\mathbf{e}_3 \tag{6}$$

[9] Note that equation (6) is still nonlinear because of the constant affine term $-g\mathbf{e}_3$.

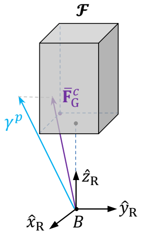
Problem Solution

Based on (5)–(6), we adopt a saturated PD control law with a feedforward term for cancelling the nonlinearity of (6) when the saturation is not activated:

$$\gamma^p = m\left(\mathbf{K}_3\left(\bar{\mathbf{r}}_{\mathrm{G}}^{\mathrm{B/G}} - \mathbf{r}_{\mathrm{G}}^{\mathrm{B/G}}\right) + \mathbf{K}_4\left(\dot{\bar{\mathbf{r}}}_{\mathrm{G}}^{\mathrm{B/G}} - \mathbf{v}_{\mathrm{G}}^{\mathrm{B/G}}\right) + g\mathbf{e}_3\right) \tag{7}$$

$$\bar{\mathbf{F}}_{\mathrm{G}}^{c} = \mathrm{sat}_{\mathcal{F}}\left(\gamma^p\right) \tag{8}$$

where $\mathbf{K}_3 \in \mathbb{R}^{3\times3}$ and $\mathbf{K}_4 \in \mathbb{R}^{3\times3}$ are the proportional and derivative gain matrices, respectively, and $\mathcal{F} \subset \mathbb{R}^3$ is a force admissible set.

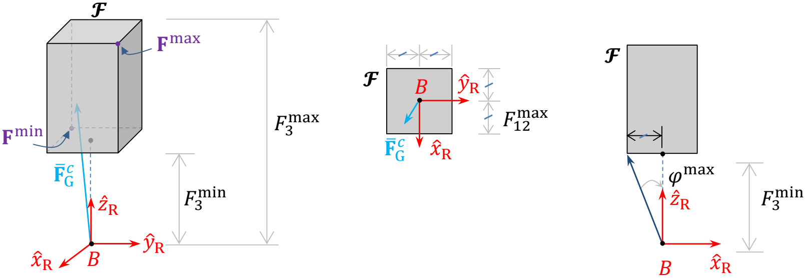


Problem Solution

The admissible set can be chosen as

$$\mathcal{F} \triangleq \left\{ \mathbf{F} \in \mathbb{R}^3 : \mathbf{F}^{\min} \preceq \mathbf{F} \preceq \mathbf{F}^{\max} \right\}$$

Denote $\mathbf{F}^{\min} = [F_1^{\min}\ F_2^{\min}\ F_3^{\min}]^{\mathrm{T}}$ and $\mathbf{F}^{\max} = [F_1^{\max}\ F_2^{\max}\ F_3^{\max}]^{\mathrm{T}}$.



Problem Solution

From the above figure,

$$F_1^{\min} = -F_1^{\max}$$
$$F_2^{\min} = -F_2^{\max}$$
$$F_1^{\max} = F_2^{\max} \triangleq F_{12}^{\max} = F_3^{\min} \tan \varphi^{\max}$$

Moreover, we can choose

$$F_3^{\min} = \frac{1}{10} mg$$
$$F_3^{\max} > 2mg$$

Problem Solution

Regarding the stability of the closed-loop translational dynamics:

1. By considering that the saturation of (8) is not activated and replacing (7)–(8) into (5)–(6) we obtain a linear time-invariant closed-loop translational model. Therefore, one can see that stability, without saturation, can be reached by choosing $\mathbf{K}_3$ and $\mathbf{K}_4$ as diagonal and positive definite.
2. For a formal stability proof considering the saturation, we suggest reading section 3.5.1 of reference [1].

Problem Solution

Control Allocation 1

To obtain $\bar{\mathbf{D}}^{\mathrm{B/R}}$, first note that its third line is the transpose of $\bar{\mathbf{n}}_{\mathrm{G}} \triangleq \bar{\mathbf{F}}^{c}_{\mathrm{G}}/\|\bar{\mathbf{F}}^{c}_{\mathrm{G}}\|$. Then, consider the formula to convert from Euler angles 123 to attitude matrix:

$$\bar{\mathbf{D}}^{\mathrm{B/R}} = \begin{bmatrix} * & * & * \\ * & * & * \\ \mathrm{s}\bar{\theta} & -\mathrm{c}\bar{\theta}\mathrm{s}\bar{\phi} & \mathrm{c}\bar{\theta}\mathrm{c}\bar{\phi} \end{bmatrix}$$

and compute $\bar{\phi}$ and $\bar{\theta}$ from

$$\bar{\phi} = -\mathrm{atan}\ n_2/n_3$$
$$\bar{\theta} = \mathrm{asin}\ n_1$$

where n_1, n_2, and n_3 are the components of $\bar{\mathbf{n}}_{\mathrm{G}}$.

Problem Solution

Finally, considering an external heading command $\bar{\psi}$, we can compute the attitude matrix command:

$$\bar{\mathbf{D}}^{\mathrm{B/R}} = \begin{bmatrix} \mathrm{c}\bar{\psi}\mathrm{c}\bar{\theta} & \mathrm{c}\bar{\psi}\mathrm{s}\bar{\theta}\mathrm{s}\bar{\phi} + \mathrm{s}\bar{\psi}\mathrm{c}\bar{\phi} & -\mathrm{c}\bar{\psi}\mathrm{s}\bar{\theta}\mathrm{c}\bar{\phi} + \mathrm{s}\bar{\psi}\mathrm{s}\bar{\phi} \\ -\mathrm{s}\psi\mathrm{c}\bar{\theta} & -\mathrm{s}\bar{\psi}\mathrm{s}\bar{\theta}\mathrm{s}\bar{\phi} + \mathrm{c}\bar{\psi}\mathrm{c}\bar{\phi} & \mathrm{s}\bar{\psi}\mathrm{s}\bar{\theta}\mathrm{c}\bar{\phi} + \mathrm{c}\bar{\psi}\mathrm{s}\bar{\phi} \\ \mathrm{s}\bar{\theta} & -\mathrm{c}\bar{\theta}\mathrm{s}\bar{\phi} & \mathrm{c}\bar{\theta}\mathrm{c}\bar{\phi} \end{bmatrix}$$

Problem Solution

Control Allocation 2

The task underlying AC-2 is to compute $\bar{\omega}_i$, $\forall i$, from $\bar{\mathbf{F}}^c_{\mathrm{G}}$ and $\bar{\mathbf{T}}^c_{\mathrm{B}}$. We are going to do it in two steps.

Step 1: Compute $\bar{f}_i$, $\forall i$, from $\bar{\mathbf{F}}^c_{\mathrm{B}}$ and $\bar{\mathbf{T}}^c_{\mathrm{B}}$ by inverting the control allocation equation [10] [11]

$$\begin{bmatrix} \bar{F}^c \\ \bar{\mathbf{T}}^c_{\mathrm{B}} \end{bmatrix} = \mathbf{\Gamma}\bar{\mathbf{f}} \tag{9}$$

where $\bar{\mathbf{f}} \triangleq \left(\bar{f}_1, \bar{f}_2, ..., \bar{f}_{n_r}\right)$.

[10] This is a general formula representing those ones obtained in Chapter 3 to relate the resulting efforts with the individual thrusts, except that the effective efforts there have been replaced here by the respective commands.

[11] In case $n_r > 4$ (and the linear system (9) has more unknowns than equations), we can use the Moore-Penrose pseudeinverse. What is the meaning of that?

Problem Solution

Step 2: Compute $\bar{\omega}_i$ from $\bar{f}_i$, $\forall i$, using the thrust model (see Section 2.6):

$$\bar{\omega}_i = \sqrt{\bar{f}_i / k_f}$$

Example: Consider a quadcopter Q+. Step 1 gives

$$\bar{\mathbf{f}} = \boldsymbol{\Xi} \begin{bmatrix} \bar{F}^c \\ \bar{\mathbf{T}}^c_{\mathrm{B}} \end{bmatrix}$$

where $\boldsymbol{\Xi} \triangleq \boldsymbol{\Gamma}^{-1}_{\mathrm{Q+}}$, while step 2 is immediate.

References . . .

References

[1] Santos, D.A., Cunha Jr, A. Flight control of a hexa-rotor airship: Uncertainty quantification for a range of temperature and pressure conditions. **ISA Transactions**, 2019.

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Thanks!