MP-282
Dynamic Modeling and Control of Multirotor Aerial Vehicles
Chapter 7: Control Allocation

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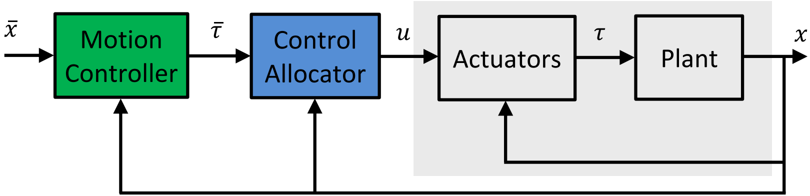
Introduction

Control Structure

A typical control system of an over-actuated mechanical plant can be described by the following block diagram:



Legend:

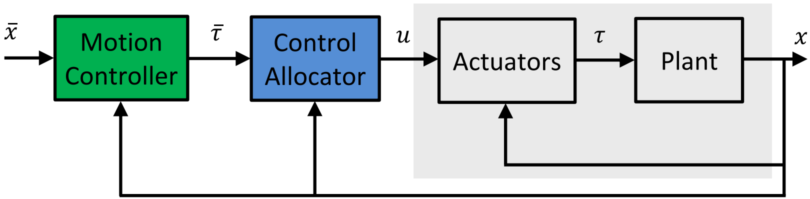
$\mathbf{x} \in \mathbb{R}^n$ - state; $\bar{\mathbf{x}} \in \mathbb{R}^n$ - state command; $\bar{\boldsymbol{\tau}} \in \mathbb{R}^m$ - virtual control [1]; $\mathbf{u} \in \mathbb{R}^p$ - actuator commands; $\boldsymbol{\tau} \in \mathbb{R}^m$ - resulting control efforts.

[1] The virtual control is typically a command for the resulting control effort.

Introduction

Comments

- The control system design can be divided into the derivation of the motion controller and control allocator.
- The main benefits of CA is achieved in control systems of over-actuated plants. Its advantages are:
  1. In case of actuator saturation/fault/failure, the control allocator can still produce the actuator commands (sometimes degraded).
  2. The actuator redundancy gives room for optimization (*e.g.*, the minimization of some cost function).



Actuator-Set Model

In general, one can model the resulting control effort $\boldsymbol{\tau} \in \mathbb{R}^m$ as

$$\boldsymbol{\tau} = \mathbf{h}(\mathbf{u}, \mathbf{x}, t)$$

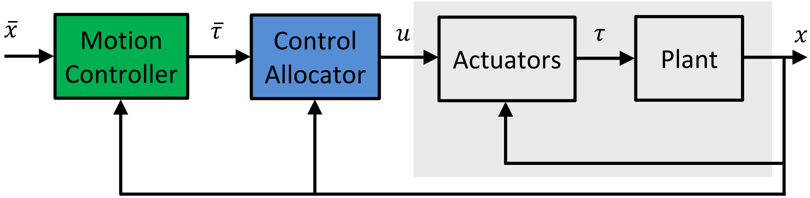
where $\mathbf{h}$ is a known map and t denotes time.

Remark

The time dependence of $\mathbf{h}$ accounts for the actuator dynamics. However, in this chapter, we will assume that the actuator dynamics are very fast and, in this case, it would suffice to write $\boldsymbol{\tau} = \mathbf{h}(\mathbf{u}, \mathbf{x})$.

Control Allocation Objective

Consider that a virtual control input $\bar{\boldsymbol{\tau}} \in \mathbb{R}^m$ is provided by the motion controller. The control allocation objective is to compute the actuator commands $\mathbf{u} \in \mathbb{R}^p$ which ensure that the resulting control effort $\boldsymbol{\tau} \in \mathbb{R}^m$ will be sufficiently close to $\bar{\boldsymbol{\tau}}$.



Introduction

Problem Formulation

It can be done as an optimization problem like

$$\min_{\mathbf{u},\mathbf{s}} \left\{ \|\mathbf{Q}\mathbf{s}\| + J(\mathbf{u}, \mathbf{x}, t) \right\}$$

$$s.t.$$

$$\bar{\boldsymbol{\tau}} - \mathbf{h}(\mathbf{u}, \mathbf{x}, t) = \mathbf{s}$$

$$\mathbf{u} \in \mathbb{U}$$

$$\mathbf{u} - \mathbf{u}_{\mathrm{prev}} \in \delta\mathbb{U}$$

where

- J is some cost function.
- $\mathbf{Q} \in \mathbb{R}^{m \times m}$ is a weighting matrix.
- $\mathbf{s} \in \mathbb{R}^{m}$ is a slack variable.
- $\mathbf{u}_{\mathrm{prev}}$ is the previous value of $\mathbf{u}$ (previous sampling time).
- $\mathbb{U}$ and $\delta\mathbb{U}$ are given compact set.

Introduction

Remarks

(1) A generic example of J:

$$J(\mathbf{x}, \mathbf{u}, t) = \frac{1}{2}(\mathbf{u} - \mathbf{u}_n)^{\mathrm{T}} \mathbf{W} (\mathbf{u} - \mathbf{u}_n)$$

where $\mathbf{W} \in \mathbb{R}^{p \times p}$ is a weighting matrix and $\mathbf{u}_n \in \mathbb{R}^p$ is the nominal value of $\mathbf{u}$. Basically, by this J, the optimization problem tries to minimize the deviation of $\mathbf{u}$ w.r.t. $\mathbf{u}_n$.

(2) Note that the cost term $\|\mathbf{Qs}\|$ along with the equality constraint $\bar{\boldsymbol{\tau}} - \mathbf{h}(\mathbf{u}, \mathbf{x}, t) = \mathbf{s}$ forces $\boldsymbol{\tau}$ towards $\bar{\boldsymbol{\tau}}$, thus contributing with the control allocation objective (see slide 7).

(3) The last two constraints represent the actuator physical bounds.

Fixed-Rotor MAVs . . .

Fixed-Rotor MAVs

Actuator-Set Model

The actuator-set models for the fixed-rotor MAVs have the common form[2]

$$\begin{bmatrix} F^c \\ \mathbf{T}_{\mathrm{B}}^c \end{bmatrix} = \mathbf{\Gamma}\mathbf{f}$$

where $\mathbf{\Gamma} \in \mathbb{R}^{4\times n_r}$ is the allocation matrix and n_r is the total number of rotors.

Assumption: The actuator dynamics are very fast, implying that $\bar{\mathbf{f}} \approx \mathbf{f}$.

Define: The virtual control input is

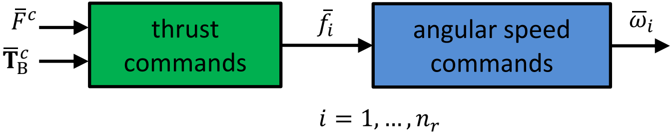
$$\bar{\boldsymbol{\tau}} \triangleq \begin{bmatrix} \bar{F}^c \\ \bar{\mathbf{T}}_{\mathrm{B}}^c \end{bmatrix}$$

[2]See Chapter 3.

Fixed-Rotor MAVs

Control Structure

We known from Chapter 6 that the control allocation for fixed-rotor MAVs can be divided into two parts:



The blue block is simply realized by the inversion of the thrust model for each individual rotor, *i.e.*,

$$\bar{\omega}_i = \sqrt{\bar{f}_i / k_f}, \quad i = 1, \dots, n_r$$

In the sequel, we are going to focus on the green block.

Fixed-Rotor MAVs

Formulation 1

Computation of thurst commands considering the rotor bounds:

$$\min_{\bar{\mathbf{f}}} \quad \|\bar{\mathbf{f}}\|^2$$
$$s.t.$$
$$\bar{\boldsymbol{\tau}} = \boldsymbol{\Gamma}\bar{\mathbf{f}}$$
$$\mathbf{e}_i^{\mathrm{T}}\bar{\mathbf{f}} \in [f_{\min}, f_{\max}], \quad \forall i = 1, ..., n_r$$
$$\mathbf{e}_i^{\mathrm{T}}(\bar{\mathbf{f}} - \bar{\mathbf{f}}_{\mathrm{prev}}) \in [\delta f_{\min}, \delta f_{\max}], \quad \forall i = 1, ..., n_r$$

where

$f_{\min} \in \mathbb{R}$ and $f_{\max} \in \mathbb{R}$ are the thrust bounds.

$\delta f_{\min} \in \mathbb{R}$ and $\delta f_{\max} \in \mathbb{R}$ are the thrust rate bounds.

$\bar{\mathbf{f}}_{\mathrm{prev}}$ is the previous value of $\bar{\mathbf{f}}$ (previous sampling time).

Remarks

(1) Note that this problem is similar to the prototype one given in slide 8, except that here we are not considering the slack variable s. Neglecting s is not an issue since we assure that the optimization input $\bar{\boldsymbol{\tau}}$ is inside its feasible set.

(2) The above problem is a quadratic program, for which there exist many efficient (comercial and free) solvers available. For solving it in MATLAB, one can use the `quadprog` command (from the Optimization Toolbox).

Fixed-Rotor MAVs

Formulation 2

One can ignore the inequality constraints in formulation 1 to obtain the following simplified problem:

$$\min_{\bar{\mathbf{f}}} \quad \|\bar{\mathbf{f}}\|^2$$
$$s.t.$$
$$\bar{\boldsymbol{\tau}} = \boldsymbol{\Gamma}\bar{\mathbf{f}}$$

The above optimization problem has a unique closed-form solution that can be obtained using Lagrange multiplier, resulting

$$\bar{\mathbf{f}}^* = \boldsymbol{\Gamma}^{\dagger}\bar{\boldsymbol{\tau}}$$

where $\boldsymbol{\Gamma}^{\dagger} \triangleq \boldsymbol{\Gamma}^{\mathrm{T}}\left(\boldsymbol{\Gamma}\boldsymbol{\Gamma}^{\mathrm{T}}\right)^{-1}$ is the Moore-Penrose's pseudo-inverse matrix.

Fixed-Rotor MAVs

After computing the optimal solution $\bar{\mathbf{f}}^*$, it is required to saturate it so as to respect the thrust and thrust-rate bounds [3] (respectively):

$$\bar{\mathbf{f}} \;\in\; \mathbb{U} \cap \left(\delta\mathbb{U} \oplus \bar{\mathbf{f}}_{\mathrm{prev}} \right)$$

Remarks

(1) The above solution can produce a dangerous mismatch between the virtual control $\bar{\boldsymbol{\tau}}$ and the resulting control effort $\boldsymbol{\tau} \triangleq (F^c, \mathbf{T}^c_{\mathrm{B}})$.

(2) However, it provides a lighter computational implementation compared with formulation 1.

[3] The symbol $\oplus$ denotes the set (or Minkowski) sum, defined as $\mathcal{A} \oplus \mathcal{B} \triangleq \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in \mathcal{A}, \mathbf{b} \in \mathcal{B}\}$.

Vectoring-Rotor MAVs ...

Vectoring-Rotor MAVs

Example 1: Quadcopter with Longitudinal-Vectoring Rotors

In Chapter 3, we obtained the actuator-set model:

$$\begin{bmatrix} F_1^c \\ F_3^c \\ \mathbf{T}_{\mathrm{B}}^c \end{bmatrix} = \mathbf{\Gamma}_{\mathrm{LV4}} \mathbf{f}$$

where

$$\mathbf{\Gamma}_{\mathrm{LV4}} \triangleq \begin{bmatrix} \mathrm{s}\beta_1 & \mathrm{s}\beta_2 & \mathrm{s}\beta_3 & \mathrm{s}\beta_4 \\ \mathrm{c}\beta_1 & \mathrm{c}\beta_2 & \mathrm{c}\beta_3 & \mathrm{c}\beta_4 \\ l\mathrm{c}\beta_1 + k\mathrm{s}\beta_1 & -l\mathrm{c}\beta_2 - k\mathrm{s}\beta_2 & -l\mathrm{c}\beta_3 + k\mathrm{s}\beta_3 & l\mathrm{c}\beta_4 - k\mathrm{s}\beta_4 \\ -l\mathrm{c}\beta_1 & -l\mathrm{c}\beta_2 & l\mathrm{c}\beta_3 & l\mathrm{c}\beta_4 \\ -l\mathrm{s}\beta_1 + k\mathrm{c}\beta_1 & l\mathrm{s}\beta_2 - k\mathrm{c}\beta_2 & l\mathrm{s}\beta_3 + k\mathrm{c}\beta_3 & -l\mathrm{s}\beta_4 - k\mathrm{c}\beta_4 \end{bmatrix}$$

Vectoring-Rotor MAVs

Consider $\beta_1 = \beta_2 = \beta_3 = \beta_4 \triangleq \beta$ and neglect the actuator dynamics. Given $\bar{\mathbf{F}}^c_{\mathrm{G}}$, $\bar{\mathbf{T}}^c_{\mathrm{B}}$, and $\mathbf{D}^{\mathrm{B/R}}$, the actuator commands $\bar{\beta}$ and $\bar{f}_i$, $i = 1, ..., 4$, are obtained by the following procedure:

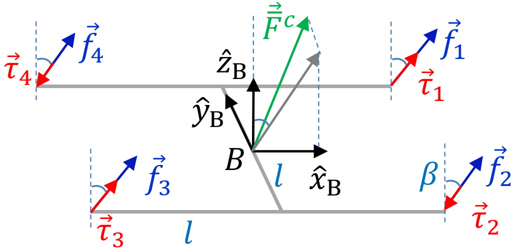
1. compute $\bar{\mathbf{F}}^c_{\mathrm{B}} \triangleq \left(\bar{F}^c_1, \bar{F}^c_2, \bar{F}^c_3\right) = \mathbf{D}^{\mathrm{B/R}} \bar{\mathbf{F}}^c_{\mathrm{G}}$
2. compute $\bar{\beta} = \mathrm{atan}\ \bar{F}^c_1 / \bar{F}^c_3$
3. compute

$$\bar{\mathbf{f}} = \left(\check{\mathbf{\Gamma}}_{\mathrm{LV4}}\right)^{-1} \begin{bmatrix} \bar{F}^c_3 \\ \bar{\mathbf{T}}^c_{\mathrm{B}} \end{bmatrix}$$

   where $\check{\mathbf{\Gamma}}_{\mathrm{LV4}}$ is obtained from $\mathbf{\Gamma}_{\mathrm{LV4}}$ by eliminating its first line.

See the figure below.

Vectoring-Rotor MAVs



On the other hand, the attitude command can be set to

$$\bar{\mathbf{D}}^{\mathrm{B/R}} = \mathbf{D}_{\bar{\mathbf{n}}_{\mathrm{B}}}(\bar{\psi})\mathbf{D}_1(-\bar{\lambda})$$

where $\bar{\psi}$ and

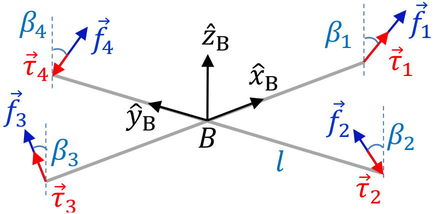
$$\bar{\lambda} \triangleq \operatorname{asin} \frac{\bar{F}_2^c}{\|\bar{\mathbf{F}}_{\mathrm{B}}^c\| \cos \bar{\beta}} \qquad \bar{\mathbf{n}}_{\mathrm{B}} \triangleq \frac{\bar{\mathbf{F}}_{\mathrm{B}}^c}{\|\bar{\mathbf{F}}_{\mathrm{B}}^c\|} \quad \square$$

Vectoring-Rotor MAVs

Exercise: Quadcopter with Transversal-Vectoring Rotors

Consider now the MAV illustrated below. Its actuator-set model was obtained in Chapter 3; it has the format:

$$\begin{bmatrix} \mathbf{F}^c_{\mathrm{B}} \\ \mathbf{T}^c_{\mathrm{B}} \end{bmatrix} = \mathbf{\Gamma}_{\mathrm{TV4}}(\beta_1, ..., \beta_4)\mathbf{f}$$



Design a control allocator for this model. Which is the corresponding attitude command?

Complementary Reading . . .

Complementary Reading

We suggest the following complementary texts:

- A survey on CA in general $\rightarrow$ Ref [1].
- CA for aerospace systems $\rightarrow$ Ref [2].
- CA for MAVs using pseudo-inverse matrix and saturation $\rightarrow$ Ref [3].

References . . .

References

[1] Johansen, T.A. and Fossen, T.I. Control Allocation - A Survey. Automatica, 49, 2013.

[2] Oppenheimer, M., Doman, D., Bolender, M. (2010). Control allocation. In W. S. Levine (Ed.), The control handbook, control system applications (2nd ed.). (Chapter 8)

[3] Ducard, G.J.J. and Hua, M.D. Discussion and Practical Aspects on Control Allocation for a Multi-Rotor Helicopter. International Archives of the Photogrammetry, Remote Sensing and Spatial Information Sciences, September 2011, Zurich.

[4] Boyd, S. and Vandenberghe, L. Convex Optimization. Cambridge University Press, 2004.

Thanks!