MP-282
Dynamic Modeling and Control of Multirotor Aerial Vehicles
Chapter 8: Flight Control Using Sliding Modes

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Stability of Nonlinear Systems . . .

Stability of Nonlinear Systems [1]

Nonlinear System

Consider a time-invariant unforced nonlinear system described by the following vectorial ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector field.

[1] For more details about Stability Theory, the reader is referred to [1].

Stability of Nonlinear Systems

Equilibrium Point

The state $\mathbf{x}^* \in \mathbb{R}^n$ is said to be an equilibrium point [2] of system (1) if once $\mathbf{x}(t)$ becomes equal to $\mathbf{x}^* \in \mathbb{R}^n$, it keeps there forever.

Therefore, the equilibrium point(s) $\mathbf{x}^*$ is (are) such that

$$\mathbf{0} = \mathbf{f}(\mathbf{x}^*) \tag{2}$$

Remark:

if $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}$, then the equilibrium points $\mathbf{x}^*$ constitute the null space of $\mathbf{A}$. Moreover, if $\mathbf{A}$ is nonsingular, this null space is a singleton, *i.e.*, there is a unique equilibrium point and it is $\mathbf{x}^* = \mathbf{0}$.

[2] Also called equilibrium state.

Preliminary Comments

- Stability is a property of an equilibrium point of the system. However, when the system has just one equilibrium point, it is common to hear that itself is stable or not.
- We consider the equilibrium point at the origin of $\mathbb{R}^n$, *i.e.*, $\mathbf{x}^* = \mathbf{0}$. There is no loss of generality here, since a change of variable can bring an arbitrary state to the origin.
- To analyze the stability of the equilibrium point $\mathbf{x}^* = \mathbf{0}$, we present the Lyapunov method. There are many possible stability concepts out there. We are going to present just the so-called (Lyapunov) stability, asymptotic stability, and exponential stability.

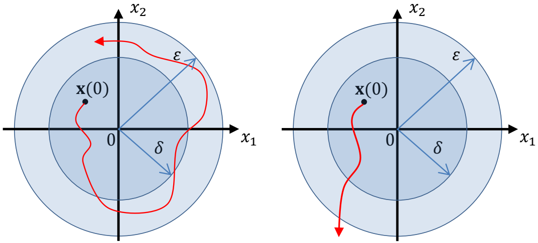
Stability of Nonlinear Systems

(Lyapunov) Stability

The equilibrium state $\mathbf{x}^* = \mathbf{0}$ is stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that [3]

$$\text{if } \|\mathbf{x}(0)\| < \delta, \text{ then } \|\mathbf{x}(t)\| < \varepsilon, \ \forall t > 0.$$

Otherwise, $\mathbf{x}^* = \mathbf{0}$ is said to be unstable.



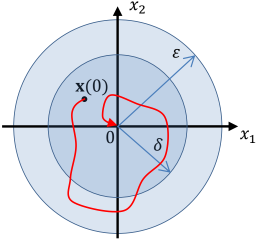
[3] Denote the zero-centered γ-ball $\mathcal{B}_\gamma \triangleq \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < \gamma\}$. Note that one could alternatively write $\mathbf{x} \in \mathcal{B}_\gamma$ in place of $\|\mathbf{x}\| < \gamma$.

Stability of Nonlinear Systems

Asymptotic Stability

The equilibrium state $\mathbf{x}^* = \mathbf{0}$ is asymptocally stable in $\mathcal{B}_\delta$ if:

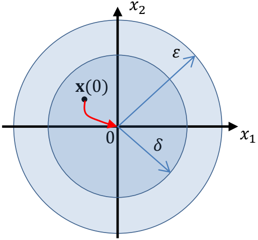
- it is stable and
- $\|\mathbf{x}(0)\| < \delta \implies \mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$.



Exponential Stability

The equilibrium state $\mathbf{x}^* = \mathbf{0}$ is exponentially stable in $\mathcal{B}_\delta$ if there exist $\alpha > 0$ and $\lambda > 0$ such that

$$\forall t > 0, \forall \mathbf{x}(0) \in \mathcal{B}_\delta, \quad \|\mathbf{x}(t)\| \leq \alpha \|\mathbf{x}(0)\| \exp(-\lambda t)$$



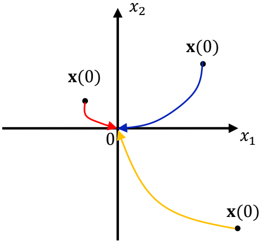
Both asymptotic and exponential stability involves convergence to the equilibrium point. The difference is that the later one specifies a convergence rate.

Stability of Nonlinear Systems

Global (Asymptotic or Exponential) Stability

The the above stability definitions characterize a system in a local neighborhood of the equilibrium point. Sometimes we need a broader definition.

If asymptotic (or exponential) stability holds for any initial condition $\mathbf{x}(0) \in \mathbb{R}^n$, the equilibrium point $\mathbf{x}^*$ is said to be global asymptotic (exponential) stable.



Stability of Nonlinear Systems

Positive Definite Function

A scalar continuous function V($\mathbf{x}$) is said to be locally positive definite if

- $V(\mathbf{0}) = 0$ and
- $V(\mathbf{x}) > 0, \forall \mathbf{x} \in \mathcal{B}_\delta - \{\mathbf{0}\}$

Moreover, if $\mathcal{B}_\delta = \mathbb{R}^n$, then $V(\mathbf{x})$ is said to be globally positive definite.

Negative Definite Function

A scalar continuous function $V(\mathbf{x})$ is said to be locally (globally) negative definite if $-V(\mathbf{x})$ is locally (globally) positive definite.

Lyapunov Function

A scalar function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a Lyapunov function of system (1) in $\mathcal{B}_\delta$ if:

- it is positive definite in $\mathcal{B}_\delta$,
- it has continuous partial derivatives in $\mathcal{B}_\delta$, and
- it is such that $\dot{V}(\mathbf{x}) \leq 0$ (along any trajectory of (1) in $\mathcal{B}_\delta$).

Stability of Nonlinear Systems [4]

Lyapunov Theorem for Local Stability

Consider that $\mathbf{x}^* = \mathbf{0}$ is an equilibrium point of system (1). If there exists a Lyapunov function V with domain $\mathcal{B}_\delta$ for system (1), then $\mathbf{x}^* = \mathbf{0}$ is locally stable. Moreover,

- if $\dot{V}(\mathbf{x}) < 0$, then $\mathbf{x}^* = \mathbf{0}$ is locally asymptotic stable and
- if there exist positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that

$$\alpha_1 \|\mathbf{x}\|^2 \leq V(\mathbf{x}) \leq \alpha_2 \|\mathbf{x}\|^2$$

$$\dot{V}(\mathbf{x}) \leq -\alpha_3 \|\mathbf{x}\|^2$$

$$\left\| \frac{\partial V}{\partial \mathbf{x}} \right\| \leq \alpha_4 \|\mathbf{x}\|$$

then $\mathbf{x}^* = 0$ is locally exponentially stable.

[4] See the proofs in (Slotine & Li, 1991).

Stability of Nonlinear Systems

Lyapunov Theorem for Global Stability

If in addition to the conditions of the above results it holds that

$$V(\mathbf{x}) \to \infty \quad \text{as} \quad \|\mathbf{x}\| \to \infty$$

then the equilibrium point $\mathbf{x}^* = \mathbf{0}$ is globally (asymptotically/exponentially) stable.

Stability of Nonlinear Systems

Finite-Time Convergence

The variable $y \in \mathbb{R}$ which satisfies the differencial equation

$$\dot{y} = -\eta y^{1/2} \tag{3}$$

converges to zero in finite time.

In fact, its solution verifies

$$y^{1/2}(t) = -\frac{\eta}{2} t + y^{1/2}(0) \tag{4}$$

from which one can compute the convergence time:

$$t_c = \frac{2}{\eta} y^{1/2}(0) \tag{5}$$

Sliding Mode Control for 2nd Order Systems . . .

Problem Definition

Consider a second-order nonlinear system described by

$$\ddot{x} = f(x, \dot{x}) + b(x, \dot{x})u + d(x, \dot{x}, u, t) \tag{6}$$

where $(x, \dot{x}) \in \mathbb{R}^2$ is the state vector, $u \in \mathbb{R}$ is the control input, and $d(x, \dot{x}, u, t) \in \mathbb{R}$ is an unkown disturbance input.

Assume that

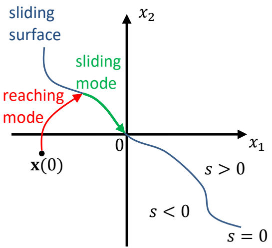
- $b(x, \dot{x}) \neq 0, \forall x, \dot{x}$
- $|d(x, \dot{x}, u, t)| \leq L > 0$, L is known

The problem is to design a control u to make $(x, \dot{x}) \to (0, 0)$ and remain there forever (in spite of the presence of the bounded disturbance $d(x, \dot{x}, u, t)$).

General Idea of the SMC

The system state is guided to the origin in two steps:

- Reaching Mode. Here, the state is guided from its initial condition to a manifold of the state space. This manifold is called sliding surface.
- Sliding Mode. Here, the state is forced to slide on the sliding surface until arriving to the origin.



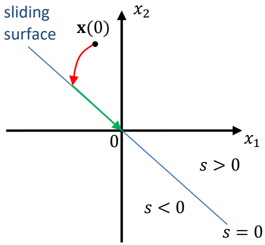
Let's construct such a control law...

Sliding Variable and Sliding Surface

Define a sliding variable $s \in \mathbb{R}$ as

$$s \triangleq c_1 x + c_2 \dot{x} \tag{7}$$

where c_1 and c_2 are scalar coefficients. The corresponding sliding surface is $\mathcal{S} \triangleq \{(x, \dot{x}) : s = 0\}$. See the illustration.



Sliding Mode

Assume that the reaching mode is so designed that once the state reaches the sliding surface, at a finite instant t_r, it keeps there forever, *i.e.*,

$$s(t) = 0, \quad \forall t \geq t_r \tag{8}$$

From (7) and (8), we see that the system dynamic in the sliding mode is described by

$$\dot{x} = -\frac{c_1}{c_2} x \tag{9}$$

Therefore, by choosing positive coefficients c_1 and c_2, we know from the linear control theory that both x and $\dot{x}$ converges to zero exponentionally.

Reaching Mode

Here we want to design u so as to drive the sliding variable s to zero in a finite time t_r. For this end, consider the Lyapunov candidate function:

$$V(s) = \frac{1}{2}s^2 \tag{10}$$

and note that the convergence $s \to 0$ (in finite time) is equivalent to $V(s) \to 0$ (in finite time). But it turns out that the later convergence can be obtained by satisfying [5]

$$\dot{V} \leq -\eta V^{1/2} \tag{11}$$

From (10)–(11), we finally obtain the so-called reaching condition:

$$s\dot{s} \leq -\frac{\eta}{\sqrt{2}}|s| \tag{12}$$

[5] See equation (3).

Control Law

Using the reaching condition (12), we can show that the control law

$$u = -\frac{1}{c_2 b(x,\dot{x})}\left(c_2 f(x,\dot{x}) + c_1\dot{x} + c_2\kappa \operatorname{sign}(s)\right) \tag{13}$$

with

$$\kappa = \eta / \left(c_2\sqrt{2}\right) + L \tag{14}$$

drives the state $(x,\dot{x})$ of system (6) to the sliding surface $\mathcal{S}$ in a finite time

$$t_r \leq \frac{\sqrt{2}}{\eta}|s(0)| \tag{15}$$

and make $(x,\dot{x}) \to (0,0)$ exponentionally.

Example

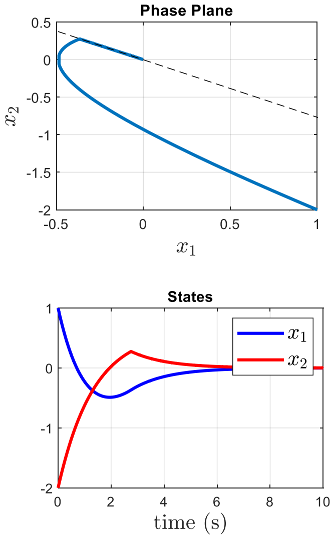
Using a MATLAB code, we simulate the closed-loop system (6) with (13), considering $f(x, \dot{x}) = x^2$, $b(x, \dot{x}) = 1$, and $d(x, \dot{x}, u, t)$ as a uniform-distributed random variable with support $[-L, L]$, where $L = 0.1$. The control parameters are set to $\eta = 2$, $c_1 = 3$, $c_2 = 4$. The system starts from the initial condition $(x, \dot{x}) = (1, -2)$.

See the plots on the next page.

Sliding Mode Control

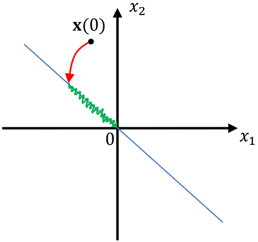
A Second-Order System

Example



Chattering

As you can see in the last plot above, the control input provided by the SMC has a high-frequency switching behavior, which, in most applications, cannot be realized in practice. Moreover, such discontinuous control can cause a zig-zag state motion across $\mathcal{S}$ (see the illustration) due to implementation imperfections, such as measurement noise, sampling. This zig-zag motion is called chattering.



Chattering Reduction by Saturation Function

Chattering can be attenuated or even eliminated by introducing a thin boundary layer, with thickness ϕ, around $\mathcal{S}$ and requiring the state $(x, \dot{x})$ to slide inside it. The state motion in the new sliding region

$$\mathcal{S}_\phi \triangleq \{(x, \dot{x}) : |s| \leq \phi\} \tag{16}$$

is called quasi-sliding mode. The corresponding quasi-sliding mode control law is obtained from (13) by replacing $\mathrm{sign}(s)$ by the saturation function

$$\mathrm{sat}(s/\phi) = \begin{cases} 1, & s > \phi \\ s/\phi, & s \in [-\phi, \phi] \\ -1, & s < -\phi \end{cases} \tag{17}$$

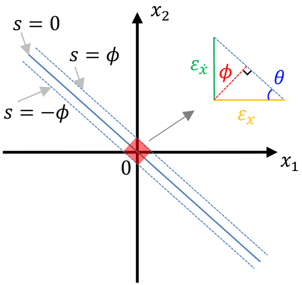
Sliding Mode Control

A Second-Order System

The figure illustrates the quasi-sliding region (or band). From its geometry, we can verify that the corresponding steady-state error is bounded by

$$\varepsilon_x = \frac{\phi}{\sin\theta} \quad \text{and} \quad \varepsilon_{\dot{x}} = \frac{c_1\phi}{c_2\sin\theta} \tag{18}$$

where $\theta \triangleq \operatorname{atan}(c_1/c_2)$.



Chattering Reduction by Sigmoid Function

Another simple way to reduce chattering is by approximating $\mathrm{sign}(s)$ by a sigmoid function, *i.e.*,

$$\mathrm{sign}(s) \approx \frac{s}{|s| + \zeta} \tag{19}$$

where ζ is a small positive scalar that must be chosen so as to trade off robustness for control smoothness.

Comments

- For a complementary reading about the basics on nonlinear systems and a specific chapter on SMC, we recommend the reference (Slotine & Li, 1991).
- The SMC literature is quite vast. Chapter 1 of (Sthessel et al., 2014) gives a nice overview of different approaches.

Sliding Mode Control for Multi-Input Systems . . .

System Model

Consider now a multi-input nonlinear system described by

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}) \tag{20}$$
$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u} + \mathbf{d}(\mathbf{x}, t) \tag{21}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state, $\mathbf{x} \triangleq (\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1 \in \mathbb{R}^{n-m}$, $\mathbf{x}_2 \in \mathbb{R}^m$, $\mathbf{u} \in \mathbb{R}^m$ is the control input, $\mathbf{d}(\mathbf{x}, t) \in \mathbb{R}^m$ is the disturbance input.

Assume that

- $\dfrac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2}\mathbf{B}(\mathbf{x})$ is nonsingular
- $\|\mathbf{d}(\mathbf{x}, t)\|_\infty \leq L > 0$, L is known

Sliding Mode Control

Multi-Input Nonlinear System

Sliding Variable

$$\mathbf{s} \triangleq \mathbf{C}_1\mathbf{x}_1 + \mathbf{C}_2\dot{\mathbf{x}}_1 \in \mathbb{R}^m \tag{22}$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are specified diagonal positive-definite matrices.

Lyapunov Candidate

$$V(\mathbf{s}) \triangleq \frac{1}{2}\mathbf{s}^{\mathrm{T}}\mathbf{s} \tag{23}$$

Sliding Condition

$$\mathbf{s}^{\mathrm{T}}\dot{\mathbf{s}} \leq -\frac{\eta}{\sqrt{2}}\|\mathbf{s}\| \tag{24}$$

Control Law

Consider that system (20)–(21) is controlled by

$$\mathbf{u} = -\left(\mathbf{C}_2 \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2} \mathbf{B}\right)^{-1} \left[\mathbf{C}_1 \mathbf{f}_1 + \mathbf{C}_2 \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1} \mathbf{f}_1 + \mathbf{C}_2 \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2} \mathbf{f}_2 + \kappa \left\| \mathbf{C}_2 \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2} \right\| \frac{\mathbf{s}}{\|\mathbf{s}\|}\right] \tag{25}$$

Therefore, we can show that the sliding mode $\mathcal{S}$ is attained if $\kappa = \eta / \left(\sqrt{2} \left\| \mathbf{C}_2 \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2} \right\|\right) + L$. Moreover, from (22), the sliding motion is described by

$$\dot{\mathbf{x}}_1 = -\mathbf{C}_2^{-1} \mathbf{C}_1 \mathbf{x}_1 \tag{26}$$

Chattering Reduction with Sigmoid Function

To reduce chattering, we can use a multivariate version of the sigmoid approximation given in equation (19):

$$\mathrm{sign}(\mathbf{s}) \triangleq \frac{\mathbf{s}}{\|\mathbf{s}\|} \approx \frac{\mathbf{s}}{\|\mathbf{s}\| + \zeta} \tag{27}$$

where ζ is a small positive scalar.

Application to MAV Flight Control . . .

Similar to Chapter 6:

- The objective here is to control the MAV position and heading, while stabilizing the 2-DOF attitude of the airframe w.r.t. the local horizontal.
- The assumptions are:
  - A1. There is a time-scale separation between the closed-loop rotational and translational dynamics.
  - A2. The actuator dynamics are negligible (another time-scale separation!).
  - A3. The actuator parameters are exactly known.
- The solution will be based on the hierarchical control arquitecture presented in Chapter 6, but the position and attitude control laws will be designed using the multi-input SMC of equation (25).

Geometric Attitude and Angular Velocity Errors

Define the attitude and angular velocity control errors

$$\tilde{\mathbf{D}} \triangleq \mathbf{D}^{\overline{\mathrm{B}}/\mathrm{B}} = \bar{\mathbf{D}}^{\mathrm{B}/\mathrm{G}} \left(\mathbf{D}^{\mathrm{B}/\mathrm{G}}\right)^{\mathrm{T}} \tag{28}$$

$$\tilde{\boldsymbol{\Omega}} \triangleq \boldsymbol{\Omega}_{\mathrm{B}}^{\overline{\mathrm{B}}/\mathrm{B}} = \tilde{\mathbf{D}}^{\mathrm{T}} \boldsymbol{\Omega}_{\overline{\mathrm{B}}}^{\overline{\mathrm{B}}/\mathrm{G}} - \boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B}/\mathrm{G}} \tag{29}$$

Further, consider the three-dimensional parameterization of $\tilde{\mathbf{D}}$ by the Gibbs vector $\tilde{\mathbf{g}} \in \mathbb{R}^3$ (see Chapter 4),

$$\tilde{\mathbf{g}} = \frac{1}{1 + \mathrm{tr}\ \tilde{\mathbf{D}}} \begin{bmatrix} \tilde{D}_{23} - \tilde{D}_{32} \\ \tilde{D}_{31} - \tilde{D}_{13} \\ \tilde{D}_{12} - \tilde{D}_{21} \end{bmatrix} \tag{30}$$

where $\tilde{D}_{ij}$ are elements of $\tilde{\mathbf{D}}$.

Attitude Error Kinematics and Dynamics

The error kinematics and dynamics can be written as

$$\dot{\tilde{\mathbf{g}}} = \frac{1}{2}\left(\tilde{\mathbf{g}}\tilde{\mathbf{g}}^{\mathrm{T}} + [\tilde{\mathbf{g}}\times] + \mathbf{I}_3\right)\tilde{\boldsymbol{\Omega}} \tag{31}$$

$$\dot{\tilde{\boldsymbol{\Omega}}} = -\mathbf{J}_{\mathrm{B}}^{-1}\left[\left(\mathbf{J}_{\mathrm{B}}\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}}\right)\times\right]\boldsymbol{\Omega}_{\mathrm{B}}^{\mathrm{B/R}} - \mathbf{J}_{\mathrm{B}}^{-1}\left(\mathbf{T}_{\mathrm{B}}^{c} + \mathbf{T}_{\mathrm{B}}^{d}\right) + \tilde{\mathbf{D}}^{\mathrm{T}}\left([\tilde{\boldsymbol{\Omega}}\times]\bar{\boldsymbol{\Omega}} + \dot{\bar{\boldsymbol{\Omega}}}\right) \tag{32}$$

where $\bar{\boldsymbol{\Omega}} \equiv \boldsymbol{\Omega}_{\overline{\mathrm{B}}}^{\overline{\mathrm{B}}/\mathrm{G}}$.

Design Model

Considering assumption A1, we have $\dot{\bar{\mathbf{\Omega}}} = \mathbf{0}$, $\bar{\mathbf{\Omega}} = \mathbf{0}$ and, consequently, $\tilde{\mathbf{\Omega}} \equiv -\mathbf{\Omega}_{\mathrm{B}}^{\mathrm{B/G}}$. Moreover, considering A2 and A3, it holds that $\bar{\mathbf{T}}_{\mathrm{B}}^{c} = \mathbf{T}_{\mathrm{B}}^{c}$. Therefore, from (31)–(32), we can obtain the following design model for the attitude controller:

$$\dot{\tilde{\mathbf{g}}} = \frac{1}{2}\left(\tilde{\mathbf{g}}\tilde{\mathbf{g}}^{\mathrm{T}} + [\tilde{\mathbf{g}}\times] + \mathbf{I}_3\right)\tilde{\mathbf{\Omega}} \tag{33}$$

$$\dot{\tilde{\mathbf{\Omega}}} = -\mathbf{J}_{\mathrm{B}}^{-1}\left[\left(\mathbf{J}_{\mathrm{B}}\tilde{\mathbf{\Omega}}\right)\times\right]\tilde{\mathbf{\Omega}} - \mathbf{J}_{\mathrm{B}}^{-1}\left(\bar{\mathbf{T}}_{\mathrm{B}}^{c} + \mathbf{T}_{\mathrm{B}}^{d}\right) \tag{34}$$

Here we add one more assumption:

A4. $\mathbf{T}_{\mathrm{B}}^{d} \in \mathbb{R}^3$ is such that $\left\|\mathbf{T}_{\mathrm{B}}^{d}\right\|_{\infty} \leq \rho_a$ and $\rho_a \in \mathbb{R}_+$ is known.

Sub-Problem 1

So the problem is to design a feedback control law for $\bar{\mathbf{T}}_{\mathrm{B}}^{c}$ that makes

$$\left(\tilde{\mathbf{g}}, \tilde{\boldsymbol{\Omega}}\right) \rightarrow (\mathbf{0}, \mathbf{0})$$

asymptotically (or exponentially).

Remark: From the Gibbs vector definition, $\tilde{\mathbf{g}} \triangleq \mathbf{a} \tan(\varphi/2)$, where **a** and φ are the principal Euler axis and angle, we see that it has singularities at the odd multiples of $\varphi = \pm\pi$ rad. Since $\tilde{\mathbf{g}}$ represents the attitude control error and, on the other hand, the singularities are far from zero, we could assume that $\tilde{\mathbf{g}}$ is sufficiently small to avoid them. In fact, the closest singularities $\pm\pi$ could be avoided by a reference governor that takes into account such state constraints.

Sliding Mode

Consider the sliding variable

$$\mathbf{s}_a \triangleq \mathbf{C}_1\tilde{\mathbf{g}} + \mathbf{C}_2\dot{\tilde{\mathbf{g}}} \tag{35}$$

where $\mathbf{C}_1 \in \mathbb{R}^3$ and $\mathbf{C}_2 \in \mathbb{R}^3$ are given positive-definite diagonal matrices.

Therefore, the sliding motion is governed by the LTI system

$$\dot{\tilde{\mathbf{g}}} = -\mathbf{C}_2^{-1}\mathbf{C}_1\tilde{\mathbf{g}} \tag{36}$$

whose origin $\tilde{\mathbf{g}} = \mathbf{0}$ is clearly exponentially stable.

Reaching Mode and Control Law

To reach the sliding surface in finite time, we need to choose $\bar{\mathbf{T}}_{\mathrm{B}}^{c}$ so as to satisfy the sliding condition:

$$\mathbf{s}_a^{\mathrm{T}}\dot{\mathbf{s}}_a \leq -\frac{\eta_a}{\sqrt{2}}\|\mathbf{s}_a\| \tag{37}$$

where $\eta_a \in \mathbb{R}_+$ is a given parameter.

For this end, we could adopt the control law

$$\bar{\mathbf{T}}_{\mathrm{B}}^{c} = -\left(\mathbf{C}_2\mathbf{M}_1\mathbf{J}_{\mathrm{B}}^{-1}\right)\Bigg(-\mathbf{C}_1\mathbf{M}_1\tilde{\boldsymbol{\Omega}} - \mathbf{C}_2\frac{d}{dt}(\mathbf{M}_1)\tilde{\boldsymbol{\Omega}} + \mathbf{C}_2\mathbf{M}_1\mathbf{J}_{\mathrm{B}}^{-1}\left[\left(\mathbf{J}_{\mathrm{B}}\tilde{\boldsymbol{\Omega}}\right)\times\right]\tilde{\boldsymbol{\Omega}} + \kappa_a\|\mathbf{C}_2\mathbf{M}_1\mathbf{J}_{\mathrm{B}}^{-1}\|\frac{\mathbf{s}_a}{\|\mathbf{s}_a\|}\Bigg) \tag{38}$$

where

$$\mathbf{M}_1 \triangleq \tilde{\mathbf{g}}\tilde{\mathbf{g}}^{\mathrm{T}} + [\tilde{\mathbf{g}}\times] + \mathbf{I}_3 \tag{39}$$

and κ_a must be chosen as

$$\kappa_a = -\rho_a - \frac{\sqrt{2}\eta_a}{\left\|\mathbf{C}_2\mathbf{M}_1\mathbf{J}_{\mathrm{B}}^{-1}\right\|} \tag{40}$$

Remark: In fact, the above control law was designed to make $\left(\tilde{\mathbf{g}}, \dot{\tilde{\mathbf{g}}}\right) \to (\mathbf{0}, \mathbf{0})$. However, from (33) we can see that a consequence of this is

$$\left(\tilde{\mathbf{g}}, \tilde{\mathbf{\Omega}}\right) \to (\mathbf{0}, \mathbf{0})$$

thus accomplishing the control objective of *Sub-Problem 1*.

Position and Velocity Errors

Define the position and velocity control errors

$$\tilde{\mathbf{r}} \triangleq \bar{\mathbf{r}}_{\mathrm{G}}^{\mathrm{B/G}} - \mathbf{r}_{\mathrm{G}}^{\mathrm{B/G}} \tag{41}$$

$$\tilde{\mathbf{v}} \triangleq \bar{\mathbf{v}}_{\mathrm{G}}^{\mathrm{B/G}} - \mathbf{v}_{\mathrm{G}}^{\mathrm{B/G}} \tag{42}$$

where $\bar{\mathbf{r}}_{\mathrm{G}}^{\mathrm{B/G}} \in \mathbb{R}^3$ and $\bar{\mathbf{v}}_{\mathrm{G}}^{\mathrm{B/G}} \in \mathbb{R}^3$ are the position and velocity commands.

Position Error Kinematics and Dynamics

The error kinematics and dynamics are described by

$$\dot{\tilde{\mathbf{r}}} = \tilde{\mathbf{v}} \tag{43}$$

$$\dot{\tilde{\mathbf{v}}} = -\frac{1}{m}\mathbf{F}^{c}_{\mathrm{G}} + g\mathbf{e}_3 - \frac{1}{m}\mathbf{F}^{d}_{\mathrm{G}} + \dot{\bar{\mathbf{v}}}^{\mathrm{B/G}}_{\mathrm{G}} \tag{44}$$

Design Model

Considering assumption A1, at this point, we have $\mathbf{D}^{\mathrm{B/G}} = \bar{\mathbf{D}}^{\mathrm{B/G}}$. Moreover, considering A2 and A3, it holds that $\bar{\mathbf{F}}^{c}_{\mathrm{B}} = \mathbf{F}^{c}_{\mathrm{B}}$. Therefore, it is also true that $\bar{\mathbf{F}}^{c}_{\mathrm{G}} = \mathbf{F}^{c}_{\mathrm{G}}$. In this context, from (43)–(44), we can obtain the following design model for the position controller:

$$\dot{\tilde{\mathbf{r}}} = \tilde{\mathbf{v}} \tag{45}$$

$$\dot{\tilde{\mathbf{v}}} = -\frac{1}{m}\bar{\mathbf{F}}^{c}_{\mathrm{G}} + g\mathbf{e}_3 - \frac{1}{m}\mathbf{F}^{d}_{\mathrm{G}} + \dot{\bar{\mathbf{v}}}^{\mathrm{B/G}}_{\mathrm{G}} \tag{46}$$

where $\mathbf{F}^{d}_{\mathrm{G}} \in \mathbb{R}^3$ is such that $\left\|\mathbf{F}^{d}_{\mathrm{G}}\right\| \leq \rho_p$ and $\rho_p \in \mathbb{R}_+$ is a known parameter.

Sub-Problem 2

So the problem is to design a feedback control law for $\bar{\mathbf{F}}^{c}_{\mathrm{G}}$ that makes

$$\left(\tilde{\mathbf{r}}, \dot{\tilde{\mathbf{r}}} \equiv \tilde{\mathbf{v}}\right) \to (\mathbf{0}, \mathbf{0})$$

asymptotically (or exponentially).

Sliding Mode

Consider the sliding variable

$$\mathbf{s}_p \triangleq \mathbf{C}_3\tilde{\mathbf{r}} + \mathbf{C}_4\tilde{\mathbf{v}} \tag{47}$$

where $\mathbf{C}_3 \in \mathbb{R}^3$ and $\mathbf{C}_4 \in \mathbb{R}^3$ are given positive-definite diagonal matrices.

Therefore, the sliding motion is governed by the LTI system

$$\dot{\tilde{\mathbf{r}}} = -\mathbf{C}_4^{-1}\mathbf{C}_3\tilde{\mathbf{r}} \tag{48}$$

whose origin $\tilde{\mathbf{r}} = \mathbf{0}$ is clearly exponentially stable.

Reaching Mode and Control Law

To reach the sliding surface in finite time, we need to choose $\bar{\mathbf{F}}_{\mathrm{G}}^{c}$ so as to satisfy the sliding condition:

$$\mathbf{s}_p^{\mathrm{T}} \dot{\mathbf{s}}_p \leq -\frac{\eta_p}{\sqrt{2}} \|\mathbf{s}_p\| \tag{49}$$

where $\eta_p \in \mathbb{R}_+$ is a given parameter.

For this end, we could adopt the control law

$$\bar{\mathbf{F}}_{\mathrm{G}}^{c} = m \mathbf{C}_4^{-1} \left(\mathbf{C}_3 \tilde{\mathbf{v}} + \mathbf{C}_4 g \mathbf{e}_3 + \mathbf{C}_4 \dot{\bar{\mathbf{v}}}_{\mathrm{G}}^{\mathrm{B/G}} - \kappa_p \frac{\|\mathbf{C}_4\|}{m} \frac{\mathbf{s}_p}{\|\mathbf{s}_p\|} \right) \tag{50}$$

where κ_p must be chosen as

$$\kappa_p = -\rho_p - \frac{m\eta_p}{\|\mathbf{C}_4\|\sqrt{2}} \tag{51}$$

Final Comments

- It remains to prove the overall stability of the afore-designed flight control system. One can try to do it using the ISS (input-to-state stability) analysis.
- The disturbance bounds ρ_a and ρ_p can be chosen to represent some modeling and/or parametric errors, besides the torque and force disturbances themselves.
- In order to enforce the time-scale separation assumption considered in the design, note that it is necessary to choose $\|\mathbf{C}_4^{-1}\mathbf{C}_3\| \gg \|\mathbf{C}_2^{-1}\mathbf{C}_1\|$ and $\eta_a \gg \eta_p$.
- It is possible to deal with control torque and force bounds in the SMC framework [6]. However, in the next chapter, we'd better treat control constraints by a reference filter (or governor).

[6] See reference [S. Ding, W. X. Zheng. Nonsingular Terminal Sliding Mode Control of Nonlinear Second-Order Systems with Input Saturation. Int. J. Rob. Nonlin. Cont., 2016.]

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