MP-208

Optimal Filtering with Aerospace Applications

Chapter 10: Navigation Using a Visual Fiducial System and Inertial Measurements

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São José dos Campos - SP
2024

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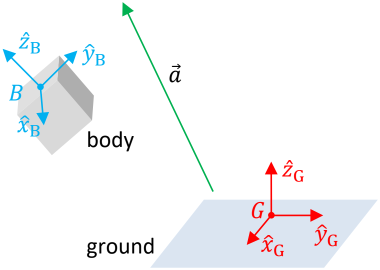
Notation and Preliminary Definitions

Geometric Vectors and Cartesian Coordinate Systems

- $\vec{a}$: geometric (or physical) vector
- $\hat{a}$: unit geometric (physical) vector
- $\mathcal{S}_{\mathrm{B}} \triangleq \{B; \hat{x}_{\mathrm{B}}, \hat{y}_{\mathrm{B}}, \hat{z}_{\mathrm{B}}\}$: body Cartesian coordinate system (CCS)
- $\mathcal{S}_{\mathrm{G}} \triangleq \{G; \hat{x}_{\mathrm{G}}, \hat{y}_{\mathrm{G}}, \hat{z}_{\mathrm{G}}\}$: ground CCS



Notation and Preliminary Definitions

Algebraic Vectors and Attitude Matrices

- $\mathbf{a}_{\mathrm{B}}$: representation of $\vec{a}$ in $\mathcal{S}_{\mathrm{B}}$ (algebraic vector); $\mathbf{a}_{\mathrm{B}} \in \mathbb{R}^3$
- $\mathbf{a}_{\mathrm{G}}$: representation of $\vec{a}$ in $\mathcal{S}_{\mathrm{G}}$ (algebraic vector); $\mathbf{a}_{\mathrm{G}} \in \mathbb{R}^3$
- $\mathbf{D}^{\mathrm{B/G}}$: attitude matrix of $\mathcal{S}_{\mathrm{B}}$ w.r.t. $\mathcal{S}_{\mathrm{G}}$; $\mathbf{D}^{\mathrm{B/G}} \in \mathrm{SO}(3)$ [1]

We can convert representations of a given geometric vector as below:

$$\mathbf{a}_{\mathrm{B}} = \mathbf{D}^{\mathrm{B/G}} \mathbf{a}_{\mathrm{G}}$$

From this and the definition of $\mathrm{SO}(3)$, we see that

$$\left(\mathbf{D}^{\mathrm{B/G}}\right)^{-1} = \left(\mathbf{D}^{\mathrm{B/G}}\right)^{\mathrm{T}} = \mathbf{D}^{\mathrm{G/B}}$$

[1] $\mathrm{SO}(3) \triangleq \{\mathbf{D} \in \mathbb{R}^{3\times 3} : \mathbf{D}\mathbf{D}^{\mathrm{T}} = \mathbf{I}_3\}$ is the special orthogonal group.

Kinematic Equations

Euler Angles

Consider the elementary rotation matrices (about axis 1,2, and 3, resp.):

$$\mathbf{D}_1(\varrho) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\varrho & \mathrm{s}\varrho \\ 0 & -\mathrm{s}\varrho & \mathrm{c}\varrho \end{bmatrix} \quad \mathbf{D}_2(\varrho) = \begin{bmatrix} \mathrm{c}\varrho & 0 & -\mathrm{s}\varrho \\ 0 & 1 & 0 \\ \mathrm{s}\varrho & 0 & \mathrm{c}\varrho \end{bmatrix}$$

$$\mathbf{D}_3(\varrho) = \begin{bmatrix} \mathrm{c}\varrho & \mathrm{s}\varrho & 0 \\ -\mathrm{s}\varrho & \mathrm{c}\varrho & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For example, considering a 1-2-3 sequence of rotations of angles denoted by ϕ, θ, and ψ, respectively, the relationship between these (Euler) angles and the attitude matrix is

$$\mathbf{D}^{\mathrm{B/G}} = \mathbf{D}_3(\psi)\mathbf{D}_2(\theta)\mathbf{D}_1(\phi)$$

Kinematic Equations

Attitude Kinematics

We can show that the attitude kinematics can be described in Euler angles 1-2-3 by

$$\dot{\boldsymbol{\alpha}}^{\mathrm{B/G}} = \mathcal{A}\left(\boldsymbol{\alpha}^{\mathrm{B/G}}\right)\boldsymbol{\omega}_{\mathrm{B}}^{\mathrm{B/R}} \tag{1}$$

where $\boldsymbol{\alpha}^{\mathrm{B/G}} \triangleq [\phi\ \theta\ \psi]^{\mathrm{T}}$, $\boldsymbol{\omega}_{\mathrm{B}}^{\mathrm{B/G}} \in \mathbb{R}^3$ is the $\mathcal{S}_{\mathrm{B}}$ representation of the angular velocity of $\mathcal{S}_{\mathrm{B}}$ w.r.t. $\mathcal{S}_{\mathrm{G}}$, and

$$\mathcal{A}\left(\boldsymbol{\alpha}^{\mathrm{B/G}}\right) \triangleq \begin{bmatrix} \mathrm{c}\psi/\mathrm{c}\theta & -\mathrm{s}\psi/\mathrm{c}\theta & 0 \\ \mathrm{s}\psi & \mathrm{c}\psi & 0 \\ -\mathrm{c}\psi\mathrm{s}\theta/\mathrm{c}\theta & \mathrm{s}\psi\mathrm{s}\theta/\mathrm{c}\theta & 1 \end{bmatrix}$$

Kinematic Equations

Position Kinematics

They are modeled by[2]

$$\frac{d}{dt_{\mathrm{G}}}\vec{r}^{\,\mathrm{B/G}} = \vec{v}^{\,\mathrm{B/G}},$$

which can be represented in $\mathcal{S}_{\mathrm{G}}$ to give

$$\dot{\mathbf{r}}_{\mathrm{G}}^{\mathrm{B/G}} = \mathbf{v}_{\mathrm{G}}^{\mathrm{B/G}} \quad (2)$$

[2]The subindex G in the time derivative is to say that it is taken with respect to an observer on S_{G}.

Sensor Modeling

Rate-Gyro:

Its measure $\breve{\boldsymbol{\omega}}_{\mathrm{B}}^{\mathrm{B/G}} \in \mathbb{R}^3$ is modeled by

$$\breve{\boldsymbol{\omega}}_{\mathrm{B}}^{\mathrm{B/G}} = \boldsymbol{\omega}_{\mathrm{B}}^{\mathrm{B/G}} + \boldsymbol{\beta}_{\mathrm{B}}^{\boldsymbol{g}} + \mathbf{w}_{\mathrm{B}}^{\boldsymbol{g}} \tag{3}$$

where $\mathbf{w}_{\mathrm{B}}^{\boldsymbol{g}} \in \mathbb{R}^3$ is a zero-mean random noise with covariance $\mathbf{Q}^{\boldsymbol{g}}$ (for simplicity, it is assumed constant and known) and $\boldsymbol{\beta}_{\mathrm{B}}^{\boldsymbol{g}} \in \mathbb{R}^3$ is a bias described by the following Wiener process:

$$\dot{\boldsymbol{\beta}}_{\mathrm{B}}^{\boldsymbol{g}} = \mathbf{w}_{\mathrm{B}}^{\beta_g} \tag{4}$$

where $\mathbf{w}_{\mathrm{B}}^{\beta_g}$ is a zero-mean random noise with covariance $\mathbf{Q}^{\beta_g}$ (it is also assumed constant and known).

Sensor Modeling

Accelerometer:

Its measure $\check{\mathbf{a}}_{\mathrm{B}}^{\mathrm{B/G}} \in \mathbb{R}^3$ is modeled by

$$\check{\mathbf{a}}_{\mathrm{B}}^{\mathrm{B/G}} = \mathbf{D}^{\mathrm{B/G}} \left(\dot{\mathbf{v}}_{\mathrm{G}}^{\mathrm{B/G}} - \mathbf{g}_{\mathrm{G}} \right) + \boldsymbol{\beta}_{\mathrm{B}}^{a} + \mathbf{w}_{\mathrm{B}}^{a} \tag{5}$$

where $\mathbf{g}_{\mathrm{G}} \triangleq -g\mathbf{e}_3$ is the gravity acceleration vector, $\mathbf{w}_{\mathrm{B}}^{a} \in \mathbb{R}^3$ is a zero-mean random noise with covariance $\mathbf{Q}^{a}$ (for simplicity, it is assumed constant and known), and $\boldsymbol{\beta}_{\mathrm{B}}^{a} \in \mathbb{R}^3$ is a bias described by the following Wiener process:

$$\dot{\boldsymbol{\beta}}_{\mathrm{B}}^{a} = \mathbf{w}_{\mathrm{B}}^{\beta_a} \tag{6}$$

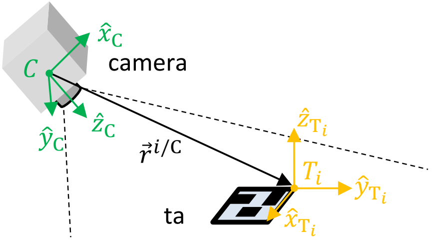
where $\mathbf{w}_{\mathrm{B}}^{\beta_a}$ is a zero-mean random noise with covariance $\mathbf{Q}^{\beta_a}$ (it is also assumed constant and known).

Visual Fiducial System:

It could be the AprilTag system, for example. Its algorithm provides indirect measures of position $\vec{r}^{i/\mathrm{C}}$ and attitude $\mathbf{D}^{i/\mathrm{C}}$ of the tag w.r.t. $\mathcal{S}_{\mathrm{C}}$. We consider just the first and assume that its measure is described by

$$\check{\mathbf{r}}_{\mathrm{C}}^{i/C} = \mathbf{r}_{\mathrm{C}}^{i/\mathrm{C}} + \mathbf{n}_{\mathrm{C}}^{i} \tag{7}$$

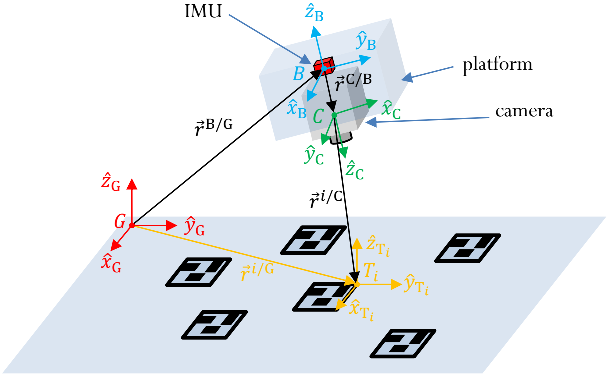
where $\mathbf{n}_{\mathrm{C}}^{i} \in \mathbb{R}^3$ is a zero-mean noise with covariance $\mathbf{R}$ (which, for simplicity, is assumed known and constant).



Problem Statement

Scenario

- The platform is a 6DOF box with a downward-facing camera and an inertial measurement unit (IMU).
- The ground has many mapped fiducial markers.

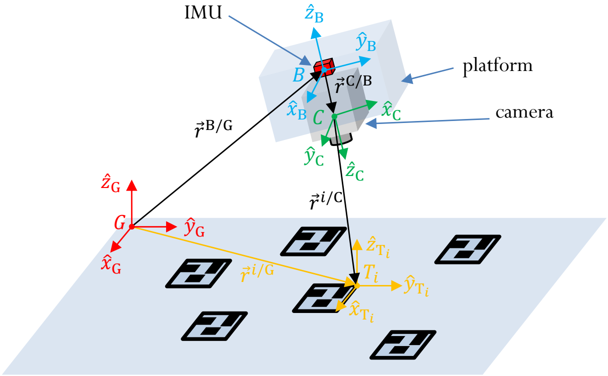


Problem Statement

Problem

It is to recursively estimate $\mathbf{r}_{\mathrm{G}}^{\mathrm{B/G}}$, $\mathbf{v}_{\mathrm{G}}^{\mathrm{B/G}}$, $\boldsymbol{\alpha}^{\mathrm{B/G}}$, $\boldsymbol{\beta}_{\mathrm{B}}^{g}$, and $\boldsymbol{\beta}_{\mathrm{B}}^{a}$ using:

- the models (1)–(7) and
- the measurements $\check{\mathbf{a}}_{\mathrm{B}}^{\mathrm{B/G}}$, $\check{\boldsymbol{\omega}}_{\mathrm{B}}^{\mathrm{B/G}}$, and $\check{\mathbf{r}}_{\mathrm{C}}^{i/\mathrm{C}}$.



Problem Solution

State Equation

The models (1)–(6) can be put together in the following state equation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) + \mathbf{G}(\mathbf{x})\,\mathbf{w}, \tag{8}$$

where

$$\mathbf{x} \triangleq \left[\left(\mathbf{r}_{\mathrm{G}}^{\mathrm{B/G}}\right)^{\mathrm{T}} \; \left(\mathbf{v}_{\mathrm{G}}^{\mathrm{B/G}}\right)^{\mathrm{T}} \; \left(\boldsymbol{\alpha}^{\mathrm{B/G}}\right)^{\mathrm{T}} \; (\boldsymbol{\beta}_{\mathrm{B}}^{a})^{\mathrm{T}} \; (\boldsymbol{\beta}_{\mathrm{B}}^{g})^{\mathrm{T}} \right]^{\mathrm{T}} \in \mathbb{R}^{15}$$

$$\mathbf{u} \triangleq \left[\left(\check{\mathbf{a}}_{\mathrm{B}}^{\mathrm{B/G}}\right)^{\mathrm{T}} \; \left(\check{\boldsymbol{\omega}}_{\mathrm{B}}^{\mathrm{B/G}}\right)^{\mathrm{T}} \right]^{\mathrm{T}} \in \mathbb{R}^{6}$$

$$\mathbf{w} \triangleq \left[(\mathbf{w}_{\mathrm{B}}^{a})^{\mathrm{T}} \; (\mathbf{w}_{\mathrm{B}}^{g})^{\mathrm{T}} \; \left(\mathbf{w}^{\beta_a}\right)^{\mathrm{T}} \; \left(\mathbf{w}^{\beta_g}\right)^{\mathrm{T}} \right]^{\mathrm{T}} \in \mathbb{R}^{12}$$

Problem Solution

State Equation (Cont.)

$$\mathbf{f}(\mathbf{x},\mathbf{u}) \triangleq \begin{bmatrix} \mathbf{v}_{\mathrm{G}}^{\mathrm{B/G}} \\ \mathbf{D}^{\mathrm{T}}\left(\boldsymbol{\alpha}^{\mathrm{B/G}}\right)\left(\check{\mathbf{a}}_{\mathrm{B}}^{\mathrm{B/G}} - \boldsymbol{\beta}_{\mathrm{B}}^{\boldsymbol{a}}\right) + \mathbf{g}_{\mathrm{G}} \\ \mathcal{A}\left(\boldsymbol{\alpha}^{\mathrm{B/G}}\right)\left(\check{\boldsymbol{\omega}}_{\mathrm{B}}^{\mathrm{B/G}} - \boldsymbol{\beta}_{\mathrm{B}}^{\boldsymbol{g}}\right) \\ \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times1} \end{bmatrix} \in \mathbb{R}^{15}$$

$$\mathbf{G}(\mathbf{x}) \triangleq \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ -\mathbf{D}^{\mathrm{T}}\left(\boldsymbol{\alpha}^{\mathrm{B/G}}\right) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & -\mathcal{A}\left(\boldsymbol{\alpha}^{\mathrm{B/G}}\right) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_3 & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_3 \end{bmatrix} \in \mathbb{R}^{15\times12}$$

Problem Solution

State Equation (Cont.)

$$\mathbf{D}\left(\boldsymbol{\alpha}^{\mathrm{B/G}}\right) = \mathbf{D}_3(\psi)\mathbf{D}_2(\theta)\mathbf{D}_1(\phi) \in \mathrm{SO}(3)$$

and

$$\phi \triangleq \mathbf{e}_1^{\mathrm{T}}\boldsymbol{\alpha}^{\mathrm{B/G}}$$
$$\theta \triangleq \mathbf{e}_2^{\mathrm{T}}\boldsymbol{\alpha}^{\mathrm{B/G}}$$
$$\psi \triangleq \mathbf{e}_3^{\mathrm{T}}\boldsymbol{\alpha}^{\mathrm{B/G}}$$

Problem Solution

Measurement Equations

Using the problem geometry (see the scenario on slide 11) and equation (7), the measurement equations can be derived as:

$$\mathbf{y}^i = \mathbf{h}^i(\mathbf{x}) + \mathbf{n}^i, \; i = 1, ..., q \tag{9}$$

where $q \in \mathbb{Z}_+$ is the number of visible markers, $\mathbf{n}^i \equiv \mathbf{n}^i_{\mathrm{C}}$, $\mathbf{y}^i \triangleq \check{\mathbf{r}}^{i/\mathrm{C}}_{\mathrm{C}}$ and

$$\mathbf{h}^i(\mathbf{x}) \triangleq \mathbf{D}^{\mathrm{C/B}} \left(\mathbf{D}\left(\boldsymbol{\alpha}^{\mathrm{B/G}}\right) \left(\mathbf{r}^{i/\mathrm{G}}_{\mathrm{G}} - \mathbf{r}^{\mathrm{B/G}}_{\mathrm{G}} \right) - \mathbf{r}^{\mathrm{C/B}}_{\mathrm{B}} \right)$$

Now we are ready for the computational exercise!

Reference

Santos, D. A. **Notas de aula de MP-282 – Dynamic Modeling and Control of Multicopters**. ITA, 2018. [Chapter 4]