MP-208
Optimal Filtering with Aerospace Applications
Section 2.2: Linear Systems

Prof. Dr. Davi Antônio dos Santos
Instituto Tecnológico de Aeronáutica
www.professordavisantos.com

São José dos Campos - SP
2023

Contents

State-Space Models...

State-Space Models

Notation:

- Denote an arbitrary discrete-time instant by $k \in \mathbb{Z}_+$.
- Denote an arbitrary continuous-time instant by $t \in \mathbb{R}_+$.
- Denote the state vectors in continuous and discrete time, respectively, by $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ and $\mathbf{x}_k \in \mathbb{R}^{n_x}$.
- Denote the control input vectors in continuous and discrete time, respectively, by $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ and $\mathbf{u}_k \in \mathbb{R}^{n_u}$.
- Denote the output vectors in continuous and discrete time, respectively, by $\mathbf{y}(t) \in \mathbb{R}^{n_y}$ and $\mathbf{y}_k \in \mathbb{R}^{n_y}$.

State-Space Model

Nonlinear Continuous-Time State-Space Model:

It consists of the state equation:

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(t, \mathbf{x}(t), \mathbf{u}(t)\right) \tag{1}$$

and of the output equation:

$$\mathbf{y}(t) = \mathbf{h}\left(t, \mathbf{x}(t), \mathbf{u}(t)\right) \tag{2}$$

where $\mathbf{f} : \mathbb{R}_+ \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and $\mathbf{h} : \mathbb{R}_+ \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_y}$ are nonlinear time-varying functions.

State-Space Model

Nonlinear Discrete-Time State-Space Model:

It consists of the state equation:

$$\mathbf{x}_{k+1} = \mathbf{f}_k\left(\mathbf{x}_k, \mathbf{u}_k\right) \tag{3}$$

and of the output equation:

$$\mathbf{y}_k = \mathbf{h}_k\left(\mathbf{x}_k, \mathbf{u}_k\right) \tag{4}$$

where $\mathbf{f}_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_x}$ and $\mathbf{h}_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_y}$ are nonlinear time-varying functions.

State-Space Model

Linear Continuous-Time State-Space Model:

It consists of the state equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{5}$$

and of the output equation:

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \tag{6}$$

where $\mathbf{A}(t) \in \mathbb{R}^{n_x \times n_x}$ is the state matrix, $\mathbf{B}(t) \in \mathbb{R}^{n_x \times n_u}$ is the input matrix, and $\mathbf{C}(t) \in \mathbb{R}^{n_y \times n_x}$ is the output matrix.

State-Space Model

Linear Discrete-Time State-Space Model:

Is consists of the state equation:

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k \tag{7}$$

and of the output equation:

$$\mathbf{y}_k = \mathbf{C}_k \mathbf{x}_k \tag{8}$$

where $\mathbf{A}_k \in \mathbb{R}^{n_x \times n_x}$ is the state matrix, $\mathbf{B}_k \in \mathbb{R}^{n_x \times n_u}$ is the input matrix, and $\mathbf{C}_k \in \mathbb{R}^{n_y \times n_x}$ is the output matrix.

State-Space Model

Continuous-Time LTI[1] State-Space Model:

It consists of the state equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{9}$$

and of the output equation:

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{10}$$

where $\mathbf{A} \in \mathbb{R}^{n_x \times n_x}$ is the state matrix, $\mathbf{B} \in \mathbb{R}^{n_x \times n_u}$ is the input matrix, and $\mathbf{C} \in \mathbb{R}^{n_y \times n_x}$ is the output matrix. Note that these matrices are constant.

[1] Linear time-invariant.

State-Space Model

Discrete-time LTI State-Space Model:

It consists of the state equation:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \tag{11}$$

and of the output equation:

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k \tag{12}$$

where $\mathbf{A} \in \mathbb{R}^{n_x \times n_x}$ is the state matrix, $\mathbf{B} \in \mathbb{R}^{n_x \times n_u}$ is the input matrix, and $\mathbf{C} \in \mathbb{R}^{n_y \times n_x}$ is the output matrix. Note that these matrices are constant.

Linearization . . .

Linearization by Taylor Series

Taylor Series Expansion:

Consider the nonlinear functions $\mathbf{f} : \mathbb{R}_+ \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and $\mathbf{h} : \mathbb{R}_+ \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_y}$ in the continuous-time nonlinear state-space model (1)-(2). Assume that they are differentiable. Their Taylor series expansion with respect to $(\mathbf{x}, \mathbf{u}) = (\bar{\mathbf{x}}, \bar{\mathbf{u}})$ are given by

$$\mathbf{f}(t, \mathbf{x}, \mathbf{u}) = \mathbf{f}(t, \bar{\mathbf{x}}, \bar{\mathbf{u}}) + \frac{d\mathbf{f}(t, \bar{\mathbf{x}}, \bar{\mathbf{u}})}{d\mathbf{x}}\delta\mathbf{x} + \frac{d\mathbf{f}(t, \bar{\mathbf{x}}, \bar{\mathbf{u}})}{d\mathbf{u}}\delta\mathbf{u} + \dots \tag{13}$$

$$\mathbf{h}(t, \mathbf{x}, \mathbf{u}) = \mathbf{h}(t, \bar{\mathbf{x}}, \bar{\mathbf{u}}) + \frac{d\mathbf{h}(t, \bar{\mathbf{x}}, \bar{\mathbf{u}})}{d\mathbf{x}}\delta\mathbf{x} + \frac{d\mathbf{h}(t, \bar{\mathbf{x}}, \bar{\mathbf{u}})}{d\mathbf{u}}\delta\mathbf{u} + \dots \tag{14}$$

where $\delta\mathbf{x} \triangleq \mathbf{x} - \bar{\mathbf{x}}$, $\delta\mathbf{u} \triangleq \mathbf{u} - \bar{\mathbf{u}}$, and the first-order derivatives are Jacobian matrices.

Linearization by Taylor Series

Linearized Model:

By truncating (13)-(14) after the first-order terms and replacing the resulting expressions into (1)-(2), we obtain the following linearized model:

$$\delta\dot{\mathbf{x}} = \mathbf{A}(t)\delta\mathbf{x} + \mathbf{B}(t)\delta\mathbf{u} \tag{15}$$

$$\delta\mathbf{y} = \mathbf{C}(t)\delta\mathbf{x} \tag{16}$$

where

$$\mathbf{A}(t) \triangleq \frac{d\mathbf{f}\left(t, \bar{\mathbf{x}}, \bar{\mathbf{u}}\right)}{d\mathbf{x}} \qquad \mathbf{B}(t) \triangleq \frac{d\mathbf{f}\left(t, \bar{\mathbf{x}}, \bar{\mathbf{u}}\right)}{d\mathbf{u}} \qquad \mathbf{C}(t) \triangleq \frac{d\mathbf{h}\left(t, \bar{\mathbf{x}}, \bar{\mathbf{u}}\right)}{d\mathbf{x}}$$

Obtaining a State-Space Model. . .

Obtaining a State-Space Model

From a Differential Equation:

Consider an nth-order ordinary differential equation in the form

$$\xi^{(n)}(t) + g\left(t, \xi(t), \dot{\xi}(t), \ddot{\xi}(t), ..., \xi^{(n-1)}(t)\right) = f(t) \tag{17}$$

where $f : \mathbb{R}_+ \to \mathbb{R}$ is a forcing function and $g : \mathbb{R}_+ \times \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbb{R}$ is a nonlinear function (in general).

Define the following state and input variables:

$$\begin{aligned} x_1 &\triangleq \xi \\ x_2 &\triangleq \dot{\xi} \\ &\vdots \\ x_n &\triangleq \xi^{(n-1)} \end{aligned} \tag{18}$$

$$u \triangleq f \tag{19}$$

...

Obtaining a State-Space Model

...

Differentiating (18) with respect to time and using (17) and (19), we can obtain the following (scalar) state equations:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_n &= -g\left(t, x_1, \ldots, x_n\right) + u
\end{aligned}
\tag{20}
$$

Obtaining a State-Space Model

From a Transfer Function:

Consider a SISO[2] system with input $u(t)$ and output $y(t)$. Denote the Laplace transforms of $u(t)$ and $y(t)$ by $U(s)$ and $Y(s)$, respectively. We call the process of obtaining a state-space model $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ from a transfer function

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} \tag{21}$$

a state-space realization.

There are infinitely many state-space realizations of the same transfer function. We adopt an example to show three canonical ones.

[2] Single Input Single Output.

Obtaining a State-Space Model

Example:

Describe the input-output relation of a system by the transfer function:

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{s+1}{s^2+5s+6} \tag{22}$$

where $U(s)$ and $Y(s)$ denote the Laplace transform of the input and output, respectively.

Obtain two distinct realizations of $G(s)$.

Solution of LTI State-Space Models. . .

Solution of LTI State-Space Models

Continuous Time:

The solution of the continuous-time LTI state equation (9) in $t \geq t_0$ is given by

$$\mathbf{x}(t) = \exp\left\{\mathbf{A}(t - t_0)\right\}\mathbf{x}(t_0) + \int_{t_0}^{t} \exp\left\{\mathbf{A}(t - \tau)\right\}\mathbf{B}\mathbf{u}(\tau)d\tau \qquad (23)$$

From the output equation (10) and equation (23), we obtain the solution of the continuous-time LTI state-space model in $t \geq t_0$ as

$$\mathbf{y}(t) = \mathbf{C}\exp\left\{\mathbf{A}(t - t_0)\right\}\mathbf{x}(t_0) + \mathbf{C}\int_{t_0}^{t} \exp\left\{\mathbf{A}(t - \tau)\right\}\mathbf{B}\mathbf{u}(\tau)d\tau \qquad (24)$$

Solution of LTI State-Space Models

Discrete Time:

The solution of the discrete-time LTI state equation (11) in $k \geq k_0$ is given by

$$\mathbf{x}_k = \mathbf{A}^{k-k_0}\mathbf{x}_{k_0} + \sum_{i=1}^{k-k_0} \mathbf{A}^{k-k_0-i}\mathbf{B}\mathbf{u}_{i+k_0-1} \tag{25}$$

From the output equation (12) and equation (25), we obtain the solution of the discrete-time LTI state-space model in $k \geq k_0$ as

$$\mathbf{y}_k = \mathbf{C}\mathbf{A}^{k-k_0}\mathbf{x}_{k_0} + \mathbf{C}\sum_{i=1}^{k-k_0} \mathbf{A}^{k-k_0-i}\mathbf{B}\mathbf{u}_{i+k_0-1} \tag{26}$$

Discretization. . .

Discretization

Time-Discretization of LTI Models:

Consider a continuous-time LTI state-space model of the form (9). Using a sampling period T and considering that $\mathbf{u}(\tau) = \mathbf{u}_k, \tau \in [t_k, t_{k+1})$, one can obtain the following discrete-time version of (9):

$$\mathbf{x}_{k+1} = \mathbf{A}_{\mathrm{d}}\mathbf{x}_k + \mathbf{B}_{\mathrm{d}}\mathbf{u}_k \tag{27}$$

$$\mathbf{y}_k = \mathbf{C}_{\mathrm{d}}\mathbf{x}_k \tag{28}$$

where

$$\mathbf{A}_{\mathrm{d}} = \exp\{\mathbf{A}T\} \qquad \mathbf{B}_{\mathrm{d}} = \int_0^T \exp\{\mathbf{A}\delta\}\, d\delta \mathbf{B} \qquad \mathbf{C}_{\mathrm{d}} = \mathbf{C}$$

This method is called zero-order hold (ZOH).

Stability ...

Stability of Continuous-Time LTI Systems

Definition:

Consider a continuous-time LTI system modeled by (9)–(10). This system is said to be stable (in the classic sense) if, for $\mathbf{u}(t) \equiv \mathbf{0}_{n_u \times 1}$,

- $\|\mathbf{y}(t)\| \leq M < \infty, \quad \forall t > t_0$
- $\lim_{t \to \infty} \|\mathbf{y}(t)\| = 0$

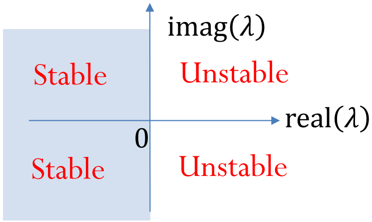
Stability of Continuous-Time LTI Systems

Stability Condition:

Consider a continuous-time LTI system modeled by (9)–(10). Denote the eigenvalues of the state matrix $\mathbf{A}$ by λ_i, $i = 1, ..., n_x$. A necessary and sufficient condition for stability (in the classic sense) of this system is

$$\text{real}\left(\lambda_i\right) < 0, \;\; i = 1, ..., n_x \tag{29}$$

Otherwise, the system is said to be unstable.



Stability of Discrete-Time LTI Systems

Definition:

Consider a discrete-time LTI system modeled by (11)–(12). This system is said to be stable (in the classic sense) if, for $\mathbf{u}_k \equiv \mathbf{0}_{n_u \times 1}$,

- $\|\mathbf{y}_k\| \leq M < \infty, \quad \forall k > k_0$
- $\lim_{k \to \infty} \|\mathbf{y}_k\| = 0$

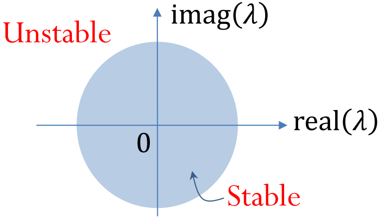
Stability of Discrete-Time LTI Systems

Stability Condition:

Consider a discrete-time LTI system modeled by (11)–(12). Denote the eigenvalues of the state matrix $\mathbf{A}$ by $\lambda_i,\ i = 1, ..., n_x$. A necessary and sufficient stability condition (in the classic sense) of this system is

$$|\lambda_i| < 1,\ i = 1, ..., n_x \tag{30}$$

Otherwise, the system is said to be unstable.



Controllability. . .

Controllability of LTI Systems

Definition:

- Consider the continuous-time LTI state-space model (9)–(10). This model is said to be controllable from $\mathbf{x}(t_0)$ if

$$\exists\ \mathbf{u}(t), t \in [t_0, t_f], t_f < \infty \tag{31}$$

  such that $\mathbf{x}(t_f) = 0$.

- If the model is controllable from any $\mathbf{x}(t_0) \in \mathbb{R}^{n_x}$, then it is said to be completely controllable.

Remark:

A similar definition can be formulated for discrete-time LTI systems modeled by (11)–(12).

Controllability of LTI Systems

Controllability Condition:

For an LTI system modeled by (9) or (11) to be completely controllable, it is necessary and sufficient that

$$\operatorname{rank}(\mathcal{C}) = n_x \tag{32}$$

where $\mathcal{C}$ is the controllability matrix defined by

$$\mathcal{C} \triangleq \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \ldots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \tag{33}$$

Observability. . .

Observability of LTI Systems

Definition:

- Consider the continuous-time LTI state-space model (9)–(10). This model is said to be observable at the point $\mathbf{x}(t_0)$ if it is possible to determine $\mathbf{x}(t_0)$ from

$$\mathbf{u}(t) \text{ and } \mathbf{y}(t), \; t \in [t_0, t_f], \tag{34}$$

  for some $t_f < \infty$.

- If the model is observable at any point $\mathbf{x}(t_0) \in \mathbb{R}^{n_x}$, then it is said to be completely observable.

Remark:

A similar definition can be formulated for discrete-time LTI systems modeled by (11)–(12).

Observabilidade de Sistemas LIT

Observability Condition:

For the LTI system described by (9)–(10) or (11)–(12) to be completely observable, it is necessary and sufficient that

$$\mathrm{rank}\left(\mathcal{O}\right) = n_x \tag{35}$$

where $\mathcal{O}$ is the observability matrix defined by

$$\mathcal{O} \triangleq \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n_x-1} \end{bmatrix} \tag{36}$$

Stabilizability and Detectability

Stabilizability:

A system is said to be stabilizable if all its uncontrollable states are stable.

Detectability:

A system is said to be detectable if all its unobservable states are stable.

References. . .

References

- Chen, C.-T. **Linear System Theory and Design**. New York: Oxford University Press, 1999.
- Ogata, K. **Engenharia de Controle Moderno**. Rio de Janeiro: LTC, 2000.