MP-208
Optimal Filtering with Aerospace Applications
Section 2.3: Set Theory

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Preliminary Definitions and Notation . . .

Preliminary Definitions and Notation

Definition:

Set is a collection of objects called elements or members.

A set can be represented by an explicit list of its elements $\zeta_i, i = 1, 2, ..., n$:

$$\mathcal{A} = \{\zeta_1, \zeta_2, ..., \zeta_n\}$$

or by the properties of its elements, *e.g.*,

$$\mathcal{B} = \{\text{all the positive integers}\}$$

Remarks:

- $\emptyset$: Empty set (which does not contain any element).
- $\mathcal{U}$: Universal set (which contains all possible elements).

Relations Between Sets

Belongs to/Element of:

If an element ζ_i belongs to a set $\mathcal{A}$, we denote $\zeta_i \in \mathcal{A}$. Otherwise, we denote $\zeta_i \notin \mathcal{A}$.

Inclusion:

A set $\mathcal{B}$ is said to be contained inside a set $\mathcal{A}$ if all the elements of $\mathcal{B}$ are also elements of $\mathcal{A}$. We denote this relation by $\mathcal{B} \subset \mathcal{A}$. We can alternatively say that $\mathcal{B}$ is a subset of $\mathcal{A}$. Otherwise, if $\mathcal{B}$ is not contained inside $\mathcal{A}$, we denote $\mathcal{B} \not\subset \mathcal{A}$.

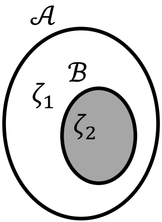
Equality:

A set $\mathcal{A}$ is said to be equal to another set $\mathcal{B}$, which is denoted by $\mathcal{A} = \mathcal{B}$, if they have the same elements.

Relations Between Sets

Venn Diagram:

It is a useful and intuitive graphical tool for representing sets, their elements, their relations, and operations. For example, in the Venn diagram below, $\zeta_1 \in \mathcal{A}$, $\zeta_2 \in \mathcal{A}$, $\zeta_2 \in \mathcal{B}$, $\zeta_1 \notin \mathcal{B}$, $\mathcal{B} \subset \mathcal{A}$, and $\mathcal{A} \not\subset \mathcal{B}$.



Operations Between Sets. . .

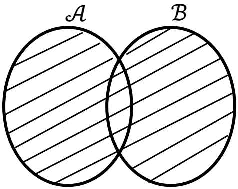
Operations Between Sets

Union:

The union (also called "sum") of two sets $\mathcal{A}$ and $\mathcal{B}$ is a third set $\mathcal{C}$ whose elements belong to $\mathcal{A}$ or to $\mathcal{B}$ or to both. We denote this operation by:

$$\mathcal{C} = \mathcal{A} \cup \mathcal{B}$$

In a Venn diagram:



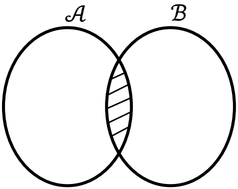
Operations Between Sets

Intersection:

The intersection (also called product) of two sets $\mathcal{A}$ and $\mathcal{B}$ is a set $\mathcal{C}$ whose elements belong to $\mathcal{A}$ and $\mathcal{B}$, simultaneously. We denote this operation by:

$$\mathcal{C} = \mathcal{A} \cap \mathcal{B}$$

In a Venn diagram:



Other Important Definitions . . .

Other Important Definitions

Mutually Exclusive Sets:

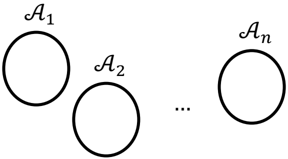
Two sets $\mathcal{A}$ and $\mathcal{B}$ are said to be mutually exclusive (or disjoint) if they have no common elements, *i.e.*,

$$\mathcal{A} \cap \mathcal{B} = \emptyset$$

Additionally, many (more than two) sets $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n$ are said to be mutually exclusive if

$$\mathcal{A}_i \cap \mathcal{A}_j = \emptyset \ , \forall i \neq j$$

In a Venn diagram:



Other Important Definitions

Partition:

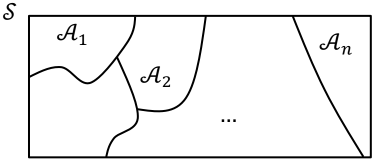
A partition $\mathcal{P}$ of a set $\mathcal{S}$ is a collection of mutually exclusive subsets $\mathcal{A}_i$, $i = 1, ..., n$, of $\mathcal{S}$ whose union is equal to $\mathcal{S}$, *i.e.*,

$$\mathcal{P} = [\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n]$$

such that

$$\mathcal{A}_1 \cup \mathcal{A}_2 \cup ... \cup \mathcal{A}_n = \mathcal{S} \quad \mathcal{A}_i \cap \mathcal{A}_j = \emptyset \ , \forall i \neq j$$

In a Venn diagram:

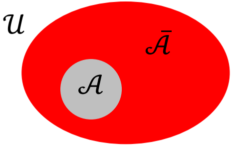


Other Important Definitions

Complement:

The complement $\bar{\mathcal{A}}$ of a set $\mathcal{A}$ is a set containing all the elements of $\mathcal{U}$ that do not belong to $\mathcal{A}$.

In a Venn diagram:



One can show that:

- $\mathcal{A} \cup \bar{\mathcal{A}} = \mathcal{U}$
- $\mathcal{A} \cap \bar{\mathcal{A}} = \emptyset$
- $\bar{\bar{\mathcal{A}}} = \mathcal{A}$
- $\bar{\mathcal{U}} = \emptyset$ and $\bar{\emptyset} = \mathcal{U}$

Other Important Definitions

Cartesian Product:

The Cartesian product between two sets $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\mathcal{A} \times \mathcal{B} \triangleq \left\{(a, b) : a \in \mathcal{A}, b \in \mathcal{B}\right\}$$

The above definition can be extended to multiple sets $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_n$ as

$$\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \triangleq \left\{(a_1, a_2, ..., a_n) : a_i \in \mathcal{A}_i, \forall i\right\}$$

Important Results. . .

Important Results

Basic Identities:

- Commutative laws: $\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$, $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$.
- Associative laws:

$$(\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} = \mathcal{A} \cup (\mathcal{B} \cup \mathcal{C})$$

$$(\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C} = \mathcal{A} \cap (\mathcal{B} \cap \mathcal{C})$$

- Distributive laws:

$$\mathcal{C} \cup (\mathcal{A} \cap \mathcal{B}) = (\mathcal{C} \cup \mathcal{A}) \cap (\mathcal{C} \cup \mathcal{B})$$

$$\mathcal{C} \cap (\mathcal{A} \cup \mathcal{B}) = (\mathcal{C} \cap \mathcal{A}) \cup (\mathcal{C} \cap \mathcal{B})$$

- Equality: $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$.

Important Results

De Morgan's Laws:

They are the following identities:

$$\overline{\mathcal{A} \cup \mathcal{B}} = \bar{\mathcal{A}} \cap \bar{\mathcal{B}}$$

$$\overline{\mathcal{A} \cap \mathcal{B}} = \bar{\mathcal{A}} \cup \bar{\mathcal{B}}$$

How to prove them?

Important Results

Duality Principle:

If in a set identity, all the unions are replaced by intersections (and vice versa) and all occurrencies of $\emptyset$ are substituted by $\mathcal{U}$ (and vice versa), the resulting identity is true.

Examples:

- Consider the identity $\mathcal{U} \cup \mathcal{A} = \mathcal{U}$. From this identity and the above result, one can immediately prove that $\emptyset \cap \mathcal{A} = \emptyset$.
- Consider the identity $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$. From this identity and the above result, one can immediately prove that $\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$.

References. . .

References

Papoulis, A.; Pillai, S. U. **Probability, Random Variables, and Stochastic Processes**. New York: McGraw-Hill, 2002.

Meyer, P. L. **Introductory Probability and Statistical Applications**. Addison-Wesley, 1970.