MP-208
Optimal Filtering with Aerospace Applications
Section 2.5: Random Variables

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Definition. . .

Definition

Random Variable:

Consider a probability space $(\Omega, \mathcal{F}, P)$. A random variable (RV) is a map $X : \Omega \to E$, where E is a set of numbers. The function X is arbitrary, except for the following restriction:

$$A(x) \triangleq \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}, \ \forall x \in E$$

$$\text{and} \quad P(A(\infty)) = 1$$

Remark:

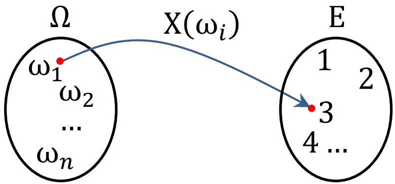
We often abbreviate:

- $\{\omega \in \Omega : X(\omega) \leq x\}$ by $\{X \leq x\}$
- $\{\omega \in \Omega : x_1 \leq X(\omega) \leq x_2\}$ by $\{x_1 \leq X \leq x_2\}$
- etc.

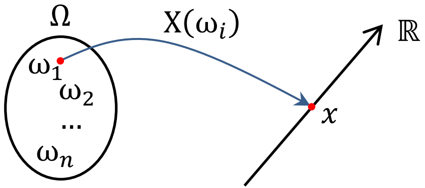
Definition

Types of Random Variables:

- **Discrete:** the set E is finite or countably infinite (*e.g.*, $E = \mathbb{N}$)



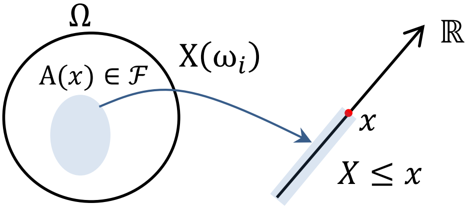
- **Continuous:** $E \subseteq \mathbb{R}$



Probability Distribution Function...

Probability Distribution Function

This course is mainly concerned with continuous random variables.



Definition:

The probability distribution function [1] of a RV X is defined by:

$$F_X(x) \triangleq P\left(A(x)\right), \quad \forall x \in \mathbb{R}$$

[1] Also known as cumulative distribution function and often abbreviated by cdf.

Probability Distribution Function

Properties:

1. $F_X(\infty) = 1$ and $F_X(-\infty) = 0$
2. If $x_1 < x_2$, then $F_X(x_1) \leq F_X(x_2)$
3. $P(\{X > x\}) = 1 - F_X(x)$
4. $P(\{x_1 < X \leq x_2\}) = F_X(x_2) - F_X(x_1)$

Remark:

We often abbreviate:

- $F_X(x)$ by $F(x)$

Probability Density Function . . .

Probability Density Function

Definition:

The probability density function (pdf) of a RV X is defined as:

$$f_X(x) \triangleq \frac{dF_X(x)}{dx}$$

We often abbreviate $f_X(x)$ by $f(x)$.

Properties:

1. $f_X(x) \geq 0$
2. $\int_{-\infty}^{+\infty} f_X(x)dx = 1$
3. $P(\{x_1 < X \leq x_2\}) = \int_{x_1}^{x_2} f_X(x)dx$

Expected Value...

Expected Value

Definition:

The expected value $E(X)$ of a RV X is defined by:

$$E(X) \triangleq \int_{-\infty}^{+\infty} x f_X(x) dx$$

Properties:

Consider a constant $a \in \mathbb{R}$ and some RVs $X_1, X_2, \ldots, X_n$.

1. If $X_1(\omega) = a, \forall \omega \in \Omega$, then $E(X_1) = a$
2. $E(aX_1) = aE(X_1)$
3. $E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$

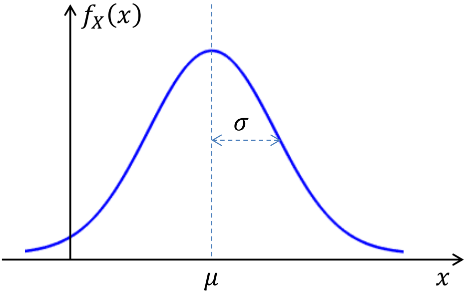
Some Standard Distributions...

Standard Distributions: Gaussian

Probability Density Function:

A RV X is said to be Gaussian (or normal) with parameters μ and σ^2 if its pdf has the form:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$



Standard Distributions: Gaussian

Probability Distribution Function:

The cdf of a Gaussian RV X is given by:

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

where

$$\Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{\xi^2}{2}\right\} d\xi$$

Notation:

To affirm that a RV X is Gaussian with parameters μ and σ^2, we use:

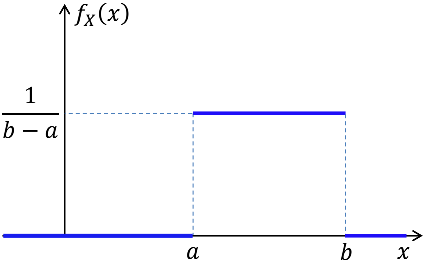
$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Standard Distributions: Uniform

Probability Density Function:

A RV X is said to be uniform in the interval (or support) $[a, b] \subset \mathbb{R}$, with $-\infty < a < b < +\infty$, if its pdf has the form:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$



Standard Distributions: Uniform

Probability Distribution Function:

The cdf of a uniform RV X is given by:

$$F_X(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases}$$

Notation:

To affirm that a RV X has uniform distribution in $[a, b]$, we write:

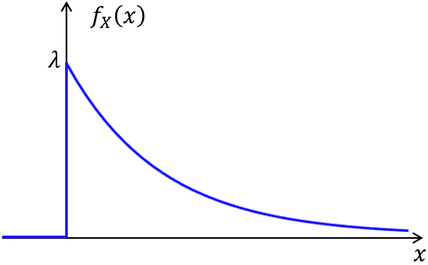
$$X \sim \mathcal{U}(a, b)$$

Standard Distributions: Exponential

Probability Density Function:

A RV X is said to be exponential with parameter $\lambda > 0$ if its pdf has the form:

$$f_X(x) = \begin{cases} \lambda \exp\{-\lambda x\}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



Standard Distributions: Exponential

Probability Distribution Function:

The cdf of an exponential RV X is given by:

$$F_X(x) = \begin{cases} 1 - \exp\{-\lambda x\}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Notation:

To affirm that a RV X is exponential with parameter λ, we use:

$$X \sim \mathcal{E}(\lambda)$$

Standard Distributions: Gamma

Probability Density Function:

A RV X is said to be gamma-distributed with parameters $\alpha > 0$ and $\beta > 0$ if its pdf has the form:

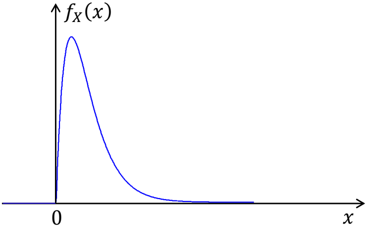
$$f_X(x) = \begin{cases} \dfrac{x^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \exp\left\{-x/\beta\right\}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

where

$$\Gamma(\alpha) \triangleq \int_0^{\infty} x^{\alpha-1} \exp\left\{-x\right\} dx$$

or

$$\Gamma(n) = (n-1)! \quad \text{se} \quad n \in \mathbb{N}$$



Standard Distributions: Gamma

Probability Distribution Function:

The cdf of a gamma-distributed RV X, for $\alpha = n$, is given by:

$$F_X(x) = \begin{cases} 1 - \sum_{k=0}^{n-1} \frac{x^k}{\beta^k k!} \exp\left\{-x/\beta\right\} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Notation:

To affirm that a RV X is gamma-distributed with parameters α and β, we use:

$$X \sim \mathcal{G}(\alpha, \beta)$$

Function of a Random Variable...

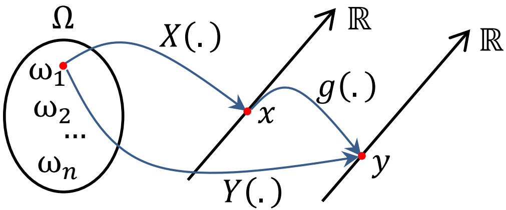
Function of a Random Variable

Consider a RV $X : \Omega \to \mathbb{R}$ and a function $g : \mathbb{R} \to \mathbb{R}$ such that:

- the domain of g contains the image of X.
- $\{\omega \in \Omega : g(X(\omega) \leq y\} \in \mathcal{F}$.

In this case,

$Y(\omega) = g(X(\omega))$ is also a random variable.



Function of a Random Variable

Theorem:

Consider a RV $X : \Omega \to \mathbb{R}$ with pdf f_X. Suppose that $g : \mathbb{R} \to \mathbb{R}$, besides satisfying the conditions given in the previous slide, be strictly monotonic and differentiable. In this case, the RV $Y = g(X)$ is characterized by the pdf

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right|$$

References. . .

References

Papoulis, A.; Pillai, S. U. **Probability, Random Variables, and Stochastic Processes**. New York: McGraw-Hill, 2002.

Ross, S. M. **A First Course in Probability**. New York: Prentice Hall, 2002.