MP-208

Optimal Filtering with Aerospace Applications

Section 2.7: Stochastic Processes

Prof. Dr. Davi Antônio dos Santos
Instituto Tecnológico de Aeronáutica
www.professordavisantos.com

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Contents

Definition. . .

Definition

Definition:

We are going to deal with continuous-state discrete-time stochastic processes (SPs). In this case, a SP is an ensemble of time sequences:

$$\left\{\mathbf{X}_k(\omega), k \in \mathbb{Z}_+, \omega \in \Omega\right\}$$

where $\mathbf{X}_k$ is a random vector (RV), *i.e.*,

$$\mathbf{X}_k : \Omega \to \mathbb{R}^n,\ \forall k \in \mathbb{Z}_+$$

Remarks:

- Common simplified notation: $\{\mathbf{X}_k\}$.
- Note that, by fixing k, $\{\mathbf{X}_k\}$ is a RV, while by fixing ω, $\{\mathbf{X}_k\}$ is a time sequence.

Definition

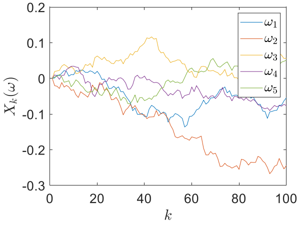
Examples:

1. Wiener Process:

$$X_k = X_{k-1} + W_{k-1}$$

$$X_1 = 0, \quad \forall \omega$$

$$W_k \sim \mathcal{N}(0, \sigma_W^2), \quad \forall k$$

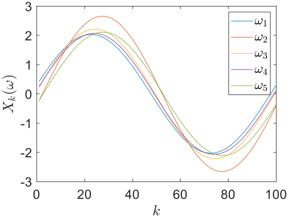


2. A parameterized SP:

$$X_k = A(\omega)\sin\left(2\pi f T k + \Phi(\omega)\right)$$

$$A \sim \mathcal{U}(a_1, a_2)$$

$$\Phi \sim \mathcal{N}(0, \sigma_\Phi^2)$$



Characterization. . .

Characterization

Definition:

The SP $\{\mathbf{X}_k\}$ is completely characterized by its joint cdf:

$$F_{\mathbf{X}_{k_1}\mathbf{X}_{k_2}\ldots\mathbf{X}_{k_n}}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$$

or, equivalently, by its joint pdf:

$$f_{\mathbf{X}_{k_1}\mathbf{X}_{k_2}\ldots\mathbf{X}_{k_n}}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$$

for any set of instants $\{k_1, k_2, \ldots, k_n\}$ and $\forall n < \infty$.

Remarks:

- Second-order cdf/pdf: $F_{\mathbf{X}_{k_1}\mathbf{X}_{k_2}}(\mathbf{x}_1, \mathbf{x}_2)$, $f_{\mathbf{X}_{k_1}\mathbf{X}_{k_2}}(\mathbf{x}_1, \mathbf{x}_2)$.
- First-order cdf/pdf: $F_{\mathbf{X}_k}(\mathbf{x})$, $f_{\mathbf{X}_k}(\mathbf{x})$.

Expectations . . .

Expectations

Mean (Function):

The mean (or mean function) $\mathbf{m}_k \in \mathbb{R}^n$ of a SP $\{\mathbf{X}_k\}$ is given by

$$\mathbf{m}_k \triangleq E(\mathbf{X}_k) = \int_{\mathbb{R}^n} \mathbf{x} f_{\mathbf{X}_k}(\mathbf{x}) d\mathbf{x}$$

Autocorrelation Function:

The autocorrelation function $\mathbf{R}_{k_1,k_2} \in \mathbb{R}^{n \times n}$ of the SP $\{\mathbf{X}_k\}$ is given by

$$\mathbf{R}_{k_1,k_2} \triangleq E\left(\mathbf{X}_{k_1} \mathbf{X}_{k_2}^{\mathrm{T}}\right) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbf{x}_1 \mathbf{x}_2^{\mathrm{T}} f_{\mathbf{X}_{k_1} \mathbf{X}_{k_2}}(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2$$

Autocovariance Function:

The autocovariance function $\mathbf{C}_{k_1,k_2} \in \mathbb{R}^{n \times n}$ of the SP $\{\mathbf{X}_k\}$ is given by

$$\mathbf{C}_{k_1,k_2} \triangleq E\left((\mathbf{X}_{k_1} - \mathbf{m}_{k_1})(\mathbf{X}_{k_2} - \mathbf{m}_{k_2})^{\mathrm{T}}\right)$$

Correlation Coefficient:

The correlation coefficient $\rho_{k_1,k_2} \in \mathbb{R}$ of the scalar SP $\{X_k\}$ is given by

$$\rho_{k_1,k_2} \triangleq \frac{C_{k_1,k_2}}{\sqrt{C_{k_1,k_1} C_{k_2,k_2}}}$$

Remarks:

- $\mathbf{C}_{k_1,k_2} = \mathbf{R}_{k_1,k_2} - \mathbf{m}_{k_1}\mathbf{m}_{k_2}^{\mathrm{T}}$.
- If $\mathbf{C}_{k_1,k_2} = \mathbf{0}, \forall k_1 \neq k_2$, then $\{\mathbf{X}_k\}$ is said to be uncorrelated.
- $\rho_{k_1,k_2} \in [-1, 1]$.

Expectations

Cross-Correlation Function:

The cross-correlation function $\mathbf{R}^{XY}_{k_1,k_2} \in \mathbb{R}^{n\times n}$ of the SPs $\{\mathbf{X}_k\}$ and $\{\mathbf{Y}_k\}$ is

$$\mathbf{R}^{XY}_{k_1,k_2} \triangleq E(\mathbf{X}_{k_1}\mathbf{Y}^{\mathrm{T}}_{k_2}) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{x}\mathbf{y}^{\mathrm{T}} f_{\mathbf{X}_{k_1}\mathbf{Y}_{k_2}}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}$$

Cross-Covariance Function:

The cross-covariance function $\mathbf{C}^{XY}_{k_1,k_2} \in \mathbb{R}^{n\times n}$ of the SPs $\{\mathbf{X}_k\}$ and $\{\mathbf{Y}_k\}$ is

$$\mathbf{C}^{XY}_{k_1,k_2} \triangleq E\left(\left(\mathbf{X}_{k_1} - \mathbf{m}^{X}_{k_1}\right)\left(\mathbf{Y}_{k_2} - \mathbf{m}^{Y}_{k_2}\right)^{\mathrm{T}}\right)$$

...

Expectations

Remarks:

- $\mathbf{C}^{XY}_{k_1,k_2} = \mathbf{R}^{XY}_{k_1,k_2} - \mathbf{m}^X_{k_1} \left(\mathbf{m}^Y_{k_2} \right)^{\mathrm{T}}$.
- If $\mathbf{C}^{XY}_{k_1,k_2} = \mathbf{0}$, $\forall k_1, k_2$, then the SPs $\{\mathbf{X}_k\}$ and $\{\mathbf{Y}_k\}$ are said to be (mutually) uncorrelated.
- if $\mathbf{R}^{XY}_{k_1,k_2} = \mathbf{0}$, $\forall k_1, k_2$, then the SPs $\{\mathbf{X}_k\}$ and $\{\mathbf{Y}_k\}$ are said to be (mutually) orthogonal.

Gaussian Stochastic Process. . .

Gaussian Stochastic Process

Definition:

The SP $\{\mathbf{X}_k\}$ is said to be Gaussian if the random variables $\mathbf{X}_{k_1}$, $\mathbf{X}_{k_2}$, ..., $\mathbf{X}_{k_n}$, $\forall n < \infty$ and any set $\{k_1, k_2, ..., k_n\}$, are jointly Gaussian.

Remark:

- A Gaussian SP $\{\mathbf{X}_k\}$ is completely characterized by its mean $\mathbf{m}_k$ and its autocorrelation $\mathbf{R}_{k_1,k_2}, \forall k_1, k_2$ (or, equivalently, by its autocovariance).

Independence. . .

Independence

Definition:

- The SP $\{\mathbf{X}_k\}$ is said to be independent if the RVs $\mathbf{X}_{k_1}$,$\mathbf{X}_{k_2}$,..., $\mathbf{X}_{k_n}$, $\forall n < \infty$ and any set $\{k_1, k_2, ..., k_n\}$, are independent.
- Two SP $\{\mathbf{X}_k\}$ and $\{\mathbf{Y}_k\}$ are said to be (mutually) independent if the RVs $\mathbf{X}_{k_1}$,$\mathbf{X}_{k_2}$,...,$\mathbf{X}_{k_n}$,$\mathbf{Y}_{k_{n+1}}$,$\mathbf{Y}_{k_{n+2}}$,...,$\mathbf{Y}_{k_{2n}}$, $\forall n < \infty$ and any set $\{k_1, ..., k_{2n}\}$, are independent.

Remarks:

- If $\{\mathbf{X}_k\}$ is independent, its joint pdf/cdf can be factored into the product of the marginal pdf/cdf.
- An independent SP is also uncorrelated. In general, the contrary is not true.
- In particular, an uncorrelated Gaussian SP is also independent.

White Noise. . .

White Noise

Definition:

The SP $\{\mathbf{X}_k\}$ is said to be a white noise (or white sequence) if it is uncorrelated, *i.e.*,

$$\mathbf{C}_{k_1,k_2} = \mathbf{0}, \ \forall k_1 \neq k_2$$

Remarks:

- Note that, in general, a white noise is not zero-mean.
- A more strong version of the above definition replaces uncorrelatedness by independence.
- Note that the autocovariance of a white noise can be written in the form:

  $$\mathbf{C}_{k_1,k_2} = \mathbf{C}_{k_1,k_1}\delta_{k_1-k_2}$$

  where δ_k is the Kronecker delta.

Stationarity . . .

Stationarity

Strict-Sense Stationarity:

The SP $\{\mathbf{X}_k\}$ is said to be strict-sense (or strongly) stationary if its joint cdf/pdf is invariant to a time shift, *i.e.*,

$$F_{\mathbf{X}_{k_1}\mathbf{X}_{k_2}\ldots\mathbf{X}_{k_n}}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = F_{\mathbf{X}_{k_1+d}\mathbf{X}_{k_2+d}\ldots\mathbf{X}_{k_n+d}}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$$
$$f_{\mathbf{X}_{k_1}\mathbf{X}_{k_2}\ldots\mathbf{X}_{k_n}}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = f_{\mathbf{X}_{k_1+d}\mathbf{X}_{k_2+d}\ldots\mathbf{X}_{k_n+d}}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$$

$\forall\ d \in \mathbb{Z}_+$.

Remarks:

- In other words, $\{\mathbf{X}_k\}$ and $\{\mathbf{X}_{k+d}\}$ have the same characterization.
- Second-order stationarity:

$$f_{\mathbf{X}_{k_1}\mathbf{X}_{k_2}}(\mathbf{x}_1, \mathbf{x}_2) = f_{\mathbf{X}_{k_1+d}\mathbf{X}_{k_2+d}}(\mathbf{x}_1, \mathbf{x}_2)$$

- First-order stationarity:

$$f_{\mathbf{X}_k}(\mathbf{x}) = f_{\mathbf{X}_{k+d}}(\mathbf{x})$$

Stationarity

Wide-Sense Stationary:

The SP $\{\mathbf{X}_k\}$ is said to be wide-sense stationary if:

$$E(\mathbf{X}_k) = \mathbf{m} \quad \text{(constant)}$$

$$E\left(\mathbf{X}_{k_1}\mathbf{X}_{k_2}^{\mathrm{T}}\right) = \mathbf{R}_\tau \quad , \quad \tau \triangleq k_1 - k_2$$

Remarks:

Consider a scalar wide-sense stationary SP $\{X_k\}$.

- Average power: $E\left(X_k^2\right) = R_0$.
- Autocovariance: $C_\tau = R_\tau - m^2$.
- Correlation coefficient: $\rho_\tau = C_\tau / C_0$.

Central Limit Theorem. . .

Central Limit Theorem

Theorem (there exist other versions):

Consider n independent RVs X_1,X_2,...,X_n. Denote $m_i = E(X_i)$ and $\sigma_i^2 = E((X_i - m_i)^2)$, $\forall i$. Now consider their sum:

$$X = X_1 + X_2 + ... + X_n$$

The mean and variance of X are, respectively,

$$m = m_1 + m_2 + ... + m_n$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + ... + \sigma_n^2$$

The Central Limit Theorem says that, under certain conditions,

$$f_X(x) \to \mathcal{N}\left(m, \sigma^2\right)$$

as $n \to \infty$.

Law of Large Numbers. . .

Law of Large Numbers

Theorem (there exist other versions):

Consider a wide-sense-stationary and uncorrelated SP $\{X_k\}$, with mean and autocovariance given by:

$$E(X_k) = m$$

$$E\left(\left(X_{k_1} - m\right)\left(X_{k_2} - m\right)\right) = \sigma^2 \delta_{k_1 - k_2}$$

Consider the sample mean:

$$\bar{X}_n \triangleq \frac{1}{n} \sum_{k=1}^{n} X_k$$

The Law of Large Numbers says that

$$\bar{X}_n \to m \quad \text{as} \quad n \to \infty \qquad (m.s.)$$

References. . .

References

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