MP-208

Optimal Filtering with Aerospace Applications

Chapter 4: Kalman Filter

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São José dos Campos - SP
2023

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Problem Definition

State Equation:

Consider the state stochastic process $\{\mathbf{X}_k\}$. Assume that its realization $\{\mathbf{x}_k\}$ is such that $\mathbf{x}_{k+1} \in \mathbb{R}^{n_x}$ is dynamically described by

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k + \mathbf{G}_k \mathbf{w}_k \tag{1}$$

where $\mathbf{u}_k \in \mathbb{R}^{n_u}$ is a known input, $\mathbf{w}_k \in \mathbb{R}^{n_w}$ is an unknown input, and $\mathbf{A}_k \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{B}_k \in \mathbb{R}^{n_x \times n_u}$, and $\mathbf{G}_k \in \mathbb{R}^{n_x \times n_w}$ are known matrices.

Assume that:

1. The initial state $\mathbf{x}_1$ is a realization of $\mathbf{X}_1 \sim \mathcal{N}(\bar{\mathbf{x}}, \bar{\mathbf{P}})$, where $\bar{\mathbf{x}} \in \mathbb{R}^{n_x}$ and $\bar{\mathbf{P}} \in \mathbb{R}^{n_x \times n_x}$ are known.
2. The sequence $\{\mathbf{w}_k\}$ is a realization of an uncorrelated SP $\{\mathbf{W}_k\}$, with $\mathbf{W}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)$, where $\mathbf{Q}_k \in \mathbb{R}^{n_w \times n_w}$ is known.
3. $\{\mathbf{W}_k\}$ and $\mathbf{X}_1$ are mutually uncorrelated.

Problem Definition

Measurement Equation:

Consider a measurement stochastic process $\{\mathbf{Y}_k\}$. Assume that its realization $\{\mathbf{y}_k\}$ is such that $\mathbf{y}_{k+1} \in \mathbb{R}^{n_y}$ is described by

$$\mathbf{y}_{k+1} = \mathbf{C}_{k+1}\mathbf{x}_{k+1} + \mathbf{v}_{k+1} \tag{2}$$

where $\mathbf{v}_{k+1} \in \mathbb{R}^{n_y}$ is an unknown input and $\mathbf{C}_{k+1} \in \mathbb{R}^{n_y \times n_x}$ is a known matrix.

Assume that:

1 The time sequence $\{\mathbf{v}_k\}$ is a realization of an uncorrelated SP $\{\mathbf{V}_k\}$, with $\mathbf{V}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$, where $\mathbf{R}_k \in \mathbb{R}^{n_y \times n_y}$ is known.

2 $\{\mathbf{V}_k\}$, $\{\mathbf{W}_k\}$, and $\mathbf{X}_1$ are mutually uncorrelated.

Problem Definition

Problem 1 (MMSE State Estimation):

The MMSE estimate $\hat{\mathbf{x}}_{k|j} \in \mathbb{R}^{n_x}$ of $\mathbf{x}_k$ from $\mathbf{y}_{1:j}$, $\mathbf{u}_{1:k-1}$, and (1)–(2) is given by

$$\hat{\mathbf{x}}_{k|j} = \arg\min_{\bar{\mathbf{x}}_k} E\left((\mathbf{X}_k - \bar{\mathbf{x}}_k)^{\mathrm{T}} (\mathbf{X}_k - \bar{\mathbf{x}}_k) \Big| \{\mathbf{Y}_{1:j} = \mathbf{y}_{1:j}\} \right) \quad (3)$$

Remarks:

1 The expectation in (3) is taken on the conditional pdf $f_{\mathbf{X}_k|\mathbf{Y}_{1:j}}(\mathbf{x}_k|\mathbf{y}_{1:j})$.

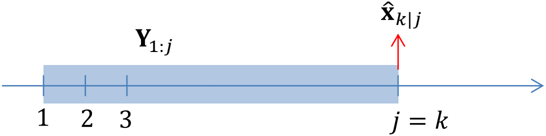
2 From Chapter 3, we known that, in general, the solution of (3) is:

$$\hat{\mathbf{x}}_{k|j} = E(\mathbf{X}_k|\mathbf{Y}_{1:j}) \quad (4)$$

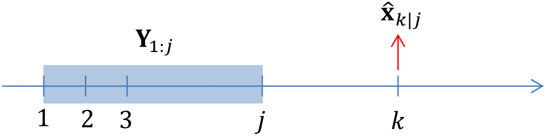
Problem Definition

3 According to the relation between k and j, we define three classes of estimation problems:

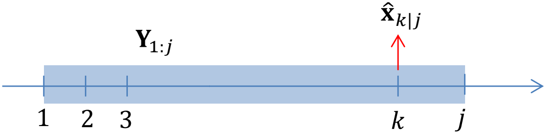
Filtering:



Prediction:



Smoothing:



Problem Solution. . .

Problem Solution

We present the explicit solution to Problem 1 for the filtering problem.

Solution Framework:

The solution is structured as a recursive algorithm, with each iteration divided into two steps:

- Prediction: the use of dynamic model (1) to obtain a predictive estimate.
- Update: The fusion of new measures with the predictive estimate to obtain a filtered (or updated) estimate.

Prediction

Problem 2 (One-Step-Ahead Prediction):

Consider that the filtered estimate at instant k as well as the corresponding conditional covariance, *i.e.*,

$$\hat{\mathbf{x}}_{k|k} \triangleq E\left(\mathbf{X}_k | \mathbf{Y}_{1:k}\right)$$

$$\mathbf{P}_{k|k} \triangleq E\left(\left(\mathbf{X}_k - \hat{\mathbf{X}}_{k|k}\right)\left(\mathbf{X}_k - \hat{\mathbf{X}}_{k|k}\right)^{\mathrm{T}} \middle| \mathbf{Y}_{1:k}\right)$$

are known. The problem is to obtain the predictive estimate at instant $k+1$, as well as the corresponding covariance:

$$\hat{\mathbf{x}}_{k+1|k} \triangleq E\left(\mathbf{X}_{k+1} | \mathbf{Y}_{1:k}\right)$$

$$\mathbf{P}_{k+1|k} \triangleq E\left(\left(\mathbf{X}_{k+1} - \hat{\mathbf{X}}_{k+1|k}\right)\left(\mathbf{X}_{k+1} - \hat{\mathbf{X}}_{k+1|k}\right)^{\mathrm{T}} \middle| \mathbf{Y}_{1:k}\right)$$

Prediction

Solution 1 (Discrete-Time Prediction):

The predictive estimate and underlying covariance are given by:

$$\hat{\mathbf{x}}_{k+1|k} = \mathbf{A}_k \hat{\mathbf{x}}_{k|k} + \mathbf{B}_k \mathbf{u}_k \quad (5)$$

$$\mathbf{P}_{k+1|k} = \mathbf{A}_k \mathbf{P}_{k|k} \mathbf{A}_k^{\mathrm{T}} + \mathbf{G}_k \mathbf{Q}_k \mathbf{G}_k^{\mathrm{T}} \quad (6)$$

We can also show that the predictive measure $\hat{\mathbf{y}}_{k+1|k} \triangleq E\left(\mathbf{Y}_{k+1} \middle| \mathbf{Y}_{1:k}\right)$ and the corresponding conditional covariance, *i.e.*,

$$\mathbf{P}^{Y}_{k+1|k} \triangleq E\left(\left(\mathbf{Y}_{k+1} - \hat{\mathbf{Y}}_{k+1|k}\right)\left(\mathbf{Y}_{k+1} - \hat{\mathbf{Y}}_{k+1|k}\right)^{\mathrm{T}} \middle| \mathbf{Y}_{1:k}\right)$$

are given by:

$$\hat{\mathbf{y}}_{k+1|k} = \mathbf{C}_{k+1} \hat{\mathbf{x}}_{k+1|k} \quad (7)$$

$$\mathbf{P}^{Y}_{k+1|k} = \mathbf{C}_{k+1} \mathbf{P}_{k+1|k} \mathbf{C}_{k+1}^{\mathrm{T}} + \mathbf{R}_{k+1} \quad (8)$$

Prediction

Finally, define the conditional cross-covariance between $\mathbf{X}_{k+1}$ and $\mathbf{Y}_{k+1}$ given $\mathbf{Y}_{1:k}$:

$$\mathbf{P}_{k+1|k}^{XY} \triangleq E\left(\left(\mathbf{X}_{k+1} - \hat{\mathbf{X}}_{k+1|k}\right)\left(\mathbf{Y}_{k+1} - \hat{\mathbf{Y}}_{k+1|k}\right)^{\mathrm{T}}\middle|\mathbf{Y}_{1:k}\right)$$

We can show that, for the problem under consideration,

$$\mathbf{P}_{k+1|k}^{XY} = \mathbf{P}_{k+1|k}\mathbf{C}_{k+1}^{\mathrm{T}} \tag{9}$$

Prediction

Solution 2 (Continuous-Time Prediction):

Suppose now that, instead of the discrete-time state equation (1), we have available a continuous-time equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t) \tag{10}$$

where $\{\mathbf{w}(t)\}$ is the realization of a continuous-time zero-mean white noise with covariance $\mathbf{Q}(t)$ at instant t.

In this case, the prediction equations are:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(t)\hat{\mathbf{x}}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{11}$$

$$\dot{\mathbf{P}}(t) = \mathbf{A}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}(t)^{\mathrm{T}} + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}(t)^{\mathrm{T}} \tag{12}$$

Prediction

Equations (11)–(12), when integrated from t_k to t_{k+1}, with initial conditions $\hat{\mathbf{x}}_{k|k}$ and $\mathbf{P}_{k|k}$, yield:

$$\hat{\mathbf{x}}_{k+1|k} = \hat{\mathbf{x}}(t_{k+1})$$
$$\mathbf{P}_{k+1|k} = \mathbf{P}(t_{k+1})$$

On the other hand, the predictive measure $\hat{\mathbf{y}}_{k+1|k}$, the corresponding conditional covariance $\mathbf{P}^{Y}_{k+1|k}$ as well as the conditional cross-covariance $\mathbf{P}^{XY}_{k+1|k}$ are all computed, as in the discrete-time prediction, by (7)–(9). ■

Remark: We commonly adopt the 4th-order Runge-Kutta method to solve the ODEs (11)–(12).

Update

Problem 3 (Update):

Consider that the predictive estimate $\hat{\mathbf{x}}_{k+1|k}$ at instant $k+1$ as well as the corresponding conditional covariance $\mathbf{P}_{k+1|k}$ are known. The problem now is to obtain the filtered estimate and the corresponding conditional covariance at instant $k+1$:

$$\hat{\mathbf{x}}_{k+1|k+1} \triangleq E\left(\mathbf{X}_{k+1} | \mathbf{Y}_{1:k+1}\right)$$

$$\mathbf{P}_{k+1|k+1} \triangleq E\left(\left(\mathbf{X}_{k+1} - \hat{\mathbf{X}}_{k+1|k+1}\right)\left(\mathbf{X}_{k+1} - \hat{\mathbf{X}}_{k+1|k+1}\right)^{\mathrm{T}} \middle| \mathbf{Y}_{1:k+1}\right)$$

Update

Solution:

The filtered estimate and the corresponding conditional covariance are given by

$$\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}\left(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k}\right) \tag{13}$$

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - \mathbf{P}^{XY}_{k+1|k}\left(\mathbf{P}^{Y}_{k+1|k}\right)^{-1}\left(\mathbf{P}^{XY}_{k+1|k}\right)^{\mathrm{T}} \tag{14}$$

where $\mathbf{K}_{k+1}$ is the Kalman gain, given by

$$\mathbf{K}_{k+1} = \mathbf{P}^{XY}_{k+1|k}\left(\mathbf{P}^{Y}_{k+1|k}\right)^{-1} \tag{15}$$

Update

...

Alternatively, equations (14)–(15) can be written, respectively, in the form:

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - \mathbf{K}_{k+1}\mathbf{C}_{k+1}\mathbf{P}_{k+1|k} \tag{16}$$

$$\mathbf{K}_{k+1} = \mathbf{P}_{k+1|k}\mathbf{C}_{k+1}^{\mathrm{T}}\left(\mathbf{C}_{k+1}\mathbf{P}_{k+1|k}\mathbf{C}_{k+1}^{\mathrm{T}} + \mathbf{R}_{k+1}\right)^{-1} \tag{17}$$

which is also very common in books and papers.

Properties

Results:

We can show the following properties of the Kalman filter:

1. The filtered estimator $\hat{\mathbf{X}}_{k|k}$ is unbiased.
2. It is the best linear state estimator (for any system).
3. It is the best estimator for linear-Guassian systems.
4. Its innovation sequence $\{\varepsilon_k\}$, where $\varepsilon_k \triangleq \mathbf{Y}_k - \hat{\mathbf{Y}}_{k|k-1}$, is Gaussian, white (uncorrelated), with zero mean, and covariance $E(\varepsilon_k \varepsilon_k^{\mathrm{T}}) \equiv \mathbf{P}^Y_{k+1|k}$.

Algorithm: Discrete Kalman Filter

% Initialization:

$\hat{\mathtt{x}}_{1|1} \leftarrow \bar{\mathtt{x}} \qquad \mathtt{P}_{1|1} \leftarrow \bar{\mathtt{P}}$

Repeat for $k > 1$:

% Prediction:

$\hat{\mathtt{x}}_{k+1|k} \leftarrow \mathtt{A}_k \hat{\mathtt{x}}_{k|k} + \mathtt{B}_k \mathtt{u}_k$

$\hat{\mathtt{y}}_{k+1|k} \leftarrow \mathtt{C}_{k+1} \hat{\mathtt{x}}_{k+1|k}$

$\mathtt{P}_{k+1|k} \leftarrow \mathtt{A}_k \mathtt{P}_{k|k} \mathtt{A}_k^{\mathrm{T}} + \mathtt{G}_k \mathtt{Q}_k \mathtt{G}_k^{\mathrm{T}}$

$\mathtt{P}^{Y}_{k+1|k} \leftarrow \mathtt{C}_{k+1} \mathtt{P}_{k+1|k} \mathtt{C}_{k+1}^{\mathrm{T}} + \mathtt{R}_{k+1}$

$\mathtt{P}^{XY}_{k+1|k} \leftarrow \mathtt{P}_{k+1|k} \mathtt{C}_{k+1}^{\mathrm{T}}$

% Update:

$\mathtt{K}_{k+1} \leftarrow \mathtt{P}^{XY}_{k+1|k} (\mathtt{P}^{Y}_{k+1|k})^{-1}$

$\hat{\mathtt{x}}_{k+1|k+1} \leftarrow \hat{\mathtt{x}}_{k+1|k} + \mathtt{K}_{k+1} (\mathtt{y}_{k+1} - \hat{\mathtt{y}}_{k+1|k})$

$\mathtt{P}_{k+1|k+1} \leftarrow \mathtt{P}_{k+1|k} - \mathtt{P}^{XY}_{k+1|k} (\mathtt{P}^{Y}_{k+1|k})^{-1} (\mathtt{P}^{XY}_{k+1|k})^{\mathrm{T}}$

end-repeat, $k \leftarrow k + 1$

Algorithm: Continuous–Discrete Kalman Filter

% Initialization:

$\hat{x}_{1|1} \leftarrow \bar{x} \qquad P_{1|1} \leftarrow \bar{P}$

Repeat for $k > 1$:

% Prediction:

Integrate from t_k to t_{k+1} with i.c. $\hat{x}_{k|k}$ and $P_{k|k}$:

$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t)$

$\dot{P}(t) = A(t)P(t) + P(t)A(t)^{T} + G(t)Q(t)G(t)^{T}$

$\hat{x}_{k+1|k} \leftarrow \hat{x}(t_{k+1}) \qquad P_{k+1|k} \leftarrow P(t_{k+1})$

$\hat{y}_{k+1|k} \leftarrow C_{k+1}\hat{x}_{k+1|k}$

$P^{Y}_{k+1|k} \leftarrow C_{k+1}P_{k+1|k}C^{T}_{k+1} + R_{k+1}$

$P^{XY}_{k+1|k} \leftarrow P_{k+1|k}C^{T}_{k+1}$

% Update:

$K_{k+1} \leftarrow P^{XY}_{k+1|k}(P^{Y}_{k+1|k})^{-1}$

$\hat{x}_{k+1|k+1} \leftarrow \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - \hat{y}_{k+1|k})$

$P_{k+1|k+1} \leftarrow P_{k+1|k} - P^{XY}_{k+1|k}(P^{Y}_{k+1|k})^{-1}(P^{XY}_{k+1|k})^{T}$

end-repeat, $k \leftarrow k + 1$

References. . .

References

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