MP-208

Optimal Filtering with Aerospace Applications

Chapter 5: Computational Aspects of the Kalman Filter

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Motivation. . .

Motivation

Numerical Difficulties of the Conventional Kalman Filter:

1. **Prediction:**

The prediction formula for the conditional covariance

$$\mathbf{P}_{k+1|k} = \mathbf{A}_k \mathbf{P}_{k|k} \mathbf{A}_k^{\mathrm{T}} + \mathbf{G}_k \mathbf{Q}_k \mathbf{G}_k^{\mathrm{T}}$$

can produce a non-symmetric matrix.

This issue can be overcome by either a suitable implementation of the products (of three matrices) or by using a square-root formulation.

Motivation

2 **Update:**

The updating formula for the conditional covariance

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - \mathbf{K}_{k+1}\mathbf{C}_{k+1}\mathbf{P}_{k+1|k}$$

can produce a non-symmetric or negative-definite or indefinite matrix.

This issue can be overcome by using the Joseph formula:

$$\mathbf{P}_{k+1|k+1} = (\mathbf{I} - \mathbf{K}_{k+1}\mathbf{C}_{k+1})\mathbf{P}_{k+1|k}(\mathbf{I} - \mathbf{K}_{k+1}\mathbf{C}_{k+1})^{\mathrm{T}} + \mathbf{K}_{k+1}\mathbf{R}_{k+1}\mathbf{K}_{k+1}^{\mathrm{T}}$$

or by using a square-root formulation.

Motivation

3 **Kalman Gain:**

The Kalman gain formula

$$\mathbf{K}_{k+1} = \mathbf{P}_{k+1|k}\mathbf{C}_{k+1}^{\mathrm{T}}\left(\mathbf{C}_{k+1}\mathbf{P}_{k+1|k}\mathbf{C}_{k+1}^{\mathrm{T}} + \mathbf{R}_{k+1}\right)^{-1}$$

has a relatively high computational cost, due to the matrix inversion, if the measure vector has a large dimension.

This issue can be mitigated by using one of the following alternatives:

- the information filter
- matrix factorization methods
- sequential update of scalar (or lower-dimensional) measures.

Information filter. . .

Information Filter

Preliminary Definitions:

Define, respectively, the updated information matrix and the predicted information matrix:

$$\mathbf{L}_{k|k} \triangleq (\mathbf{P}_{k|k})^{-1} \tag{1}$$

$$\mathbf{L}_{k+1|k} \triangleq (\mathbf{P}_{k+1|k})^{-1} \tag{2}$$

Define also the following transformed updated and predicted estimates, respectively:

$$\hat{\mathbf{z}}_{k|k} \triangleq \mathbf{L}_{k|k}\hat{\mathbf{x}}_{k|k} \tag{3}$$

$$\hat{\mathbf{z}}_{k+1|k} \triangleq \mathbf{L}_{k+1|k}\hat{\mathbf{x}}_{k+1|k} \tag{4}$$

Information Filter

Prediction:

Given $\mathbf{L}_{k|k}$ and $\hat{\mathbf{z}}_{k|k}$, one can calculate $\mathbf{L}_{k+1|k}$ and $\hat{\mathbf{z}}_{k+1|k}$ by:

$$\boldsymbol{\Pi}_k = \mathbf{A}_k^{-\mathrm{T}} \mathbf{L}_{k|k} \mathbf{A}_k^{-1} \tag{5}$$

$$\mathbf{K}_k^* = \boldsymbol{\Pi}_k \mathbf{G}_k \left(\mathbf{G}_k^{\mathrm{T}} \boldsymbol{\Pi}_k \mathbf{G}_k + \mathbf{Q}_k^{-1} \right)^{-1} \tag{6}$$

$$\hat{\mathbf{z}}_{k+1|k} = \left(\mathbf{I} - \mathbf{K}_k^* \mathbf{G}_k^{\mathrm{T}} \right) \mathbf{A}_k^{-\mathrm{T}} \hat{\mathbf{z}}_{k|k} + \left(\mathbf{I} - \mathbf{K}_k^* \mathbf{G}_k^{\mathrm{T}} \right) \boldsymbol{\Pi}_k \mathbf{B}_k \mathbf{u}_k \tag{7}$$

$$\mathbf{L}_{k+1|k} = \left(\mathbf{I} - \mathbf{K}_k^* \mathbf{G}_k^{\mathrm{T}} \right) \boldsymbol{\Pi}_k \tag{8}$$

Remark:

Note that $\mathbf{A}_k$ must be non-singular!

Information Filter

Update:

Given $\mathbf{L}_{k+1|k}$ and $\hat{\mathbf{z}}_{k+1|k}$, one can calculate $\mathbf{L}_{k+1|k+1}$ and $\hat{\mathbf{z}}_{k+1|k+1}$ by:

$$\hat{\mathbf{z}}_{k+1|k+1} = \hat{\mathbf{z}}_{k+1|k} + \mathbf{C}_{k+1}^{\mathrm{T}} \mathbf{R}_{k+1}^{-1} \mathbf{y}_{k+1} \quad (9)$$

$$\mathbf{L}_{k+1|k+1} = \mathbf{L}_{k+1|k} + \mathbf{C}_{k+1}^{\mathrm{T}} \mathbf{R}_{k+1}^{-1} \mathbf{C}_{k+1} \quad (10)$$

Whenever necessary, one can recover the filtered estimate as well as the corresponding covariance, by:

$$\mathbf{P}_{k+1|k+1} = \mathbf{L}_{k+1|k+1}^{-1}$$

$$\hat{\mathbf{x}}_{k+1|k+1} = \mathbf{P}_{k+1|k+1} \hat{\mathbf{z}}_{k+1|k+1}$$

Filter with sequential update. . .

Filter with Sequential Update

Problem Definition:

We want now to update the prior estimate $\hat{\mathbf{x}}_{k+1|k} \in \mathbb{R}^{n_x}$ by assimilating the scalar components of $\mathbf{y}_{k+1} \in \mathbb{R}^{n_y}$,

$$\mathbf{y}_{k+1} \triangleq \left[y_{k+1,1} \; y_{k+1,2} \; \cdots \; y_{k+1,n_y} \right]^{\mathrm{T}},$$

one by one, sequentially.

Filter with Sequential Update

Problem Solution (for uncorrelated measurement noise):

First, let us describe $Y_{k+1,i}$ by:

$$Y_{k+1,i} = \mathbf{C}_{k+1,i}\mathbf{X}_{k+1} + V_{k+1,i} \tag{11}$$

where $\mathbf{C}_{k+1,i} \in \mathbb{R}^{1\times n_x}$ is a known matrix, $\mathbf{X}_{k+1} \in \mathbb{R}^{n_x}$ is the state vector, and $V_{k+1,i} \in \mathbb{R}$ is the measurement noise, with $V_{k+1,i} \sim \mathcal{N}(0, R_{k+1,i})$, and $R_{k+1,i} \in \mathbb{R}$.

Moreover, one can suppose that the set $\left\{\mathbf{X}_{k+1}, \mathbf{V}_{1:k}, V_{k+1,1}, ..., V_{k+1,i}\right\}$ is uncorrelated and the conditional distribution of $\mathbf{X}_{k+1}$ given

$$\left\{\mathbf{Y}_{1:k}, Y_{k+1,1}, \ldots, Y_{k+1,i-1}\right\}$$

is

$$\mathbf{X}_{k+1}|\mathbf{Y}_{1:k}, Y_{k+1,1}, ..., Y_{k+1,i-1} \sim \mathcal{N}(\hat{\mathbf{x}}_{k+1|k+1,i-1}, \mathbf{P}_{k+1|k+1,i-1}) \tag{12}$$

Filter with Sequential Update

...Problem Solution:

The optimal (in the MMSE sense) assimilation of the realization $y_{k+1,i}$ of $Y_{k+1,i}$ to the prior estimate $\hat{\mathbf{x}}_{k+1|k+1,i-1}$ is given by

$$\mathbf{K}_{k+1,i} = \mathbf{P}_{k+1|k+1,i-1}\mathbf{C}_{k+1,i}^{\mathrm{T}} / \left(\mathbf{C}_{k+1,i}\mathbf{P}_{k+1|k+1,i-1}\mathbf{C}_{k+1,i}^{\mathrm{T}} + R_{k+1,i}\right) \quad (13)$$

$$\hat{\mathbf{x}}_{k+1|k+1,i} = \hat{\mathbf{x}}_{k+1|k+1,i-1} + \mathbf{K}_{k+1,i}\left(y_{k+1,i} - \mathbf{C}_{k+1,i}\hat{\mathbf{x}}_{k+1|k+1,i-1}\right) \quad (14)$$

$$\mathbf{P}_{k+1|k+1,i} = \mathbf{P}_{k+1|k+1,i-1} - \mathbf{K}_{k+1,i}\mathbf{C}_{k+1,i}\mathbf{P}_{k+1|k+1,i-1} \quad (15)$$

Filter with Sequential Update

At the beginning of the loop:

When $i = 1$, the prior information of equation (12) is reduced to

$$\mathbf{X}_{k+1} | \mathbf{Y}_{1:k} \sim \mathcal{N}(\hat{\mathbf{x}}_{k+1|k}, \mathbf{P}_{k+1|k}) \tag{16}$$

and, therefore,

$$\hat{\mathbf{x}}_{k+1|k+1,0} = \hat{\mathbf{x}}_{k+1|k} \tag{17}$$

$$\mathbf{P}_{k+1|k+1,0} = \mathbf{P}_{k+1|k} \tag{18}$$

At the end of the loop:

After assimilating the n_y-th scalar measure, the filtered estimate as well as the corresponding covariance are obtained as

$$\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k+1,n_y} \tag{19}$$

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k+1,n_y} \tag{20}$$

Filter with Sequential Update

Correlated Measurement Noise:

Suppose now that the covariance $\mathbf{R}_{k+1} \in \mathbb{R}^{n_y \times n_y}$ is not diagonal (*i.e.*, its components are, in general, correlated). However, it turns out that one can obtain a transformed measurement model:

$$\bar{\mathbf{Y}}_{k+1} = \bar{\mathbf{C}}_{k+1}\mathbf{X}_{k+1} + \bar{\mathbf{V}}_{k+1} \tag{21}$$

where $\bar{\mathbf{Y}}_{k+1} = \mathbf{R}_{k+1}^{-1/2}\mathbf{Y}_{k+1}$, $\bar{\mathbf{C}}_{k+1} = \mathbf{R}_{k+1}^{-1/2}\mathbf{C}_{k+1}$, $\bar{\mathbf{V}}_{k+1} = \mathbf{R}_{k+1}^{-1/2}\mathbf{V}_{k+1}$, and $\mathbf{R}_{k+1}^{-1/2}$ is the Cholesky factor of $\mathbf{R}_{k+1}$, such that:

$$\bar{\mathbf{R}}_{k+1} \triangleq E\left(\bar{\mathbf{V}}_{k+1}\bar{\mathbf{V}}_{k+1}^{\mathrm{T}}\right) = \mathbf{I}_m \tag{22}$$

In this case, in (13)–(15), we use $\bar{\mathbf{y}}_{k+1}$, $\bar{\mathbf{C}}_{k+1}$, and $\bar{\mathbf{R}}_{k+1}$ instead of $\mathbf{y}_{k+1}$, $\mathbf{C}_{k+1}$, and $\mathbf{R}_{k+1}$, respectively.

Square-root filter . . .

Square-Root Filter

Preliminary Definitions:

Consider the Cholesky decomposition of $\mathbf{P}_{k+1|k}$ and $\mathbf{P}_{k+1|k+1}$:

$$\mathbf{P}_{k+1|k} = \mathbf{S}_{k+1|k}\mathbf{S}_{k+1|k}^{\mathrm{T}} \tag{23}$$

$$\mathbf{P}_{k+1|k+1} = \mathbf{S}_{k+1|k+1}\mathbf{S}_{k+1|k+1}^{\mathrm{T}} \tag{24}$$

where the Cholesky factors $\mathbf{S}_{k+1|k} \in \mathbb{R}^{n_x \times n_x}$ and $\mathbf{S}_{k+1|k+1} \in \mathbb{R}^{n_x \times n_x}$ are lower-triangular matrices.

Denote the Cholesky factors of $\mathbf{Q}_k$ and $\mathbf{R}_k$ by $\mathbf{Q}_k^{1/2}$ and $\mathbf{R}_k^{1/2}$, respectively.

Square-Root Filter

Prediction:

Consider the time-propagation formula for the conditional state covariance:

$$\mathbf{P}_{k+1|k} = \mathbf{A}_k \mathbf{P}_{k|k} \mathbf{A}_k^{\mathrm{T}} + \mathbf{G}_k \mathbf{Q}_k \mathbf{G}_k^{\mathrm{T}} \tag{25}$$

Using the definitions (23)–(24), we can immediately re-write (25) into the form

$$\mathbf{S}_{k+1|k} \mathbf{S}_{k+1|k}^{\mathrm{T}} = \mathbf{M}\mathbf{M}^{\mathrm{T}} \tag{26}$$

$$\mathbf{M} \triangleq \begin{bmatrix} \mathbf{A}_k \mathbf{S}_{k|k} & \mathbf{G}_k \mathbf{Q}_k^{1/2} \end{bmatrix} \tag{27}$$

We can finally obtain $\mathbf{S}_{k+1|k} = \mathcal{R}^{\mathrm{T}}$, where $\mathcal{R}$ is the upper-triangular matrix of the QR decomposition of $\mathbf{M}^{\mathrm{T}}$.

Square-Root Filter

Update:

Consider the Joseph formula for the state covariance update:

$$\mathbf{P}_{k+1|k+1} = (\mathbf{I} - \mathbf{K}_{k+1}\mathbf{C}_{k+1})\mathbf{P}_{k+1|k}(\mathbf{I} - \mathbf{K}_{k+1}\mathbf{C}_{k+1})^{\mathrm{T}} + \mathbf{K}_{k+1}\mathbf{R}_{k+1}\mathbf{K}_{k+1}^{\mathrm{T}} \tag{28}$$

Again, using the definitions (23)–(24), we can immediately re-write (28) into the form

$$\mathbf{S}_{k+1|k+1}\mathbf{S}_{k+1|k+1}^{\mathrm{T}} = \bar{\mathbf{M}}\bar{\mathbf{M}}^{\mathrm{T}} \tag{29}$$

$$\bar{\mathbf{M}} \triangleq \begin{bmatrix} (\mathbf{I} - \mathbf{K}_{k+1}\mathbf{C}_{k+1})\mathbf{S}_{k+1|k} & \mathbf{K}_{k+1}\mathbf{R}_{k+1}^{1/2} \end{bmatrix} \tag{30}$$

We can finally obtain $\mathbf{S}_{k+1|k+1} = \bar{\mathcal{R}}^{\mathrm{T}}$, where $\bar{\mathcal{R}}$ is the upper-triangular matrix of the QR decomposition of $\bar{\mathbf{M}}^{\mathrm{T}}$.

Square-Root Filter

Kalman Gain:

Consider the traditional Kalman gain formula:

$$\mathbf{K}_{k+1} = \mathbf{P}^{XY}_{k+1|k} \left(\mathbf{P}^{Y}_{k+1|k}\right)^{-1} \tag{31}$$

where

$$\mathbf{P}^{Y}_{k+1|k} = \mathbf{C}_{k+1}\mathbf{P}_{k+1|k}\mathbf{C}^{\mathrm{T}}_{k+1} + \mathbf{R}_{k+1}$$

$$\mathbf{P}^{XY}_{k+1|k} = \mathbf{P}_{k+1|k}\mathbf{C}^{\mathrm{T}}_{k+1}$$

Consider also the Cholesky decomposition of $\mathbf{P}^{Y}_{k+1|k} = \check{\mathbf{M}}\check{\mathbf{M}}^{\mathrm{T}}$. It turns out that we can obtain $\mathbf{K}_{k+1}$ by solving the system of equations

$$\mathbf{K}_{k+1}\check{\mathbf{M}}\check{\mathbf{M}}^{\mathrm{T}} = \mathbf{P}^{XY}_{k+1|k} \tag{32}$$

by backward- and forward-substitution, respectively.

References . . .

References

- Bar-Shalom, Y., Li, R. X., Kirubarajan, T. **Estimation with Applications to Tracking and Navigation**. New Jersey: Jonh Wiley & Sons, 2001.
- Anderson, B. D. O., Moore, J. B. **Optimal Filtering**. New York: Dover, 2005.
- Bierman, G. J. **Factorization Methods for Discrete Sequential Estimation**. New York: Academic Press, 1977.