MP-208
Optimal Filtering with Aerospace Applications
Chapter 6: Extended Kalman Filter
Part I: Discrete-Time Formulation

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Problem Definition. . .

Problem Definition

State Equation:

Consider a state SP $\{\mathbf{X}_k\}$ and its realization $\{\mathbf{x}_k\}$, with $\mathbf{x}_k \in \mathbb{R}^{n_x}$ dynamically described by

$$\mathbf{x}_{k+1} = \mathbf{f}_k\left(\mathbf{x}_k, \mathbf{u}_k\right) + \mathbf{G}_k \mathbf{w}_k \quad (1)$$

where $\mathbf{u}_k \in \mathbb{R}^{n_u}$ is a known input, $\mathbf{w}_k \in \mathbb{R}^{n_w}$ is an unknown input, $\mathbf{f}_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ is a given non-linear function, and $\mathbf{G}_k \in \mathbb{R}^{n_x \times n_w}$ is a known matrix.

Assume that:

1. The initial state $\mathbf{x}_1$ is a realization of $\mathbf{X}_1$, which is assumed to be approximately symmetric and to have a known mean $\bar{\mathbf{x}} \in \mathbb{R}^{n_x}$ and a known covariance $\bar{\mathbf{P}} \in \mathbb{R}^{n_x \times n_x}$. For short, we denote $\mathbf{X}_1 \sim (\bar{\mathbf{x}}, \bar{\mathbf{P}})$.
2. The sequence $\{\mathbf{w}_k\}$ is a realization of an uncorrelated SP $\{\mathbf{W}_k\}$, with an approx. symmetric $\mathbf{W}_k \sim (\mathbf{0}, \mathbf{Q}_k)$, where $\mathbf{Q}_k \in \mathbb{R}^{n_w \times n_w}$ is known.
3. $\left\{\{\mathbf{W}_k\}, \mathbf{X}_1\right\}$ is uncorrelated.

Problem Definition

Measurement Equation:

Consider a measurement SP $\{\mathbf{Y}_k\}$ and its realization $\{\mathbf{y}_k\}$, where $\mathbf{y}_{k+1} \in \mathbb{R}^{n_y}$ is described by

$$\mathbf{y}_{k+1} = \mathbf{h}_{k+1}(\mathbf{x}_{k+1}) + \mathbf{v}_{k+1} \tag{2}$$

where $\mathbf{v}_{k+1} \in \mathbb{R}^{n_y}$ is an unknown input and $\mathbf{h}_{k+1} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ is a given non-linear function.

Assume that:

1. The sequence $\{\mathbf{v}_k\}$ is a realization of the uncorrelated SP $\{\mathbf{V}_k\}$, with approx. symmetric $\mathbf{V}_k \sim (\mathbf{0}, \mathbf{R}_k)$, where $\mathbf{R}_k \in \mathbb{R}^{n_y \times n_y}$ is known.
2. $\{\{\mathbf{V}_k\}, \{\mathbf{W}_k\}, \mathbf{X}_1\}$ is uncorrelated.

Problem Definition

Problem Statement:

The problem is to obtain an approximately optimal (MMSE) recursive filter for estimating $\{\mathbf{x}_k\}$ using $\{\mathbf{y}_k\}$, $\{\mathbf{u}_k\}$, and (1)–(2).

Comments:

In this course, we are going to present the following solutions to this problem:

i. EKF: Extended Kalman Filter (in this chapter);
ii. UKF: Unscented Kalman Filter; and
iii. EnKF: Ensemble Kalman Filter.

Discrete Extended Kalman Filter...

Discrete Extended Kalman Filter

Functional Approximations:

Let us approximate the non-linear functions $\mathbf{f}_k$ and $\mathbf{h}_k$ by Taylor series expansion and truncation:

$$\mathbf{f}_k\left(\mathbf{x}_k, \mathbf{u}_k\right) \approx \mathbf{f}_k\left(\hat{\mathbf{x}}_{k|k}, \mathbf{u}_k\right) + \mathbf{F}_k\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\right) \quad (3)$$

$$\mathbf{h}_{k+1}\left(\mathbf{x}_{k+1}\right) \approx \mathbf{h}_{k+1}\left(\hat{\mathbf{x}}_{k+1|k}\right) + \mathbf{H}_{k+1}\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right) \quad (4)$$

where $\hat{\mathbf{x}}_{k|k} \triangleq E\left(\mathbf{x}_k|\mathbf{Y}_{1:k}\right)$, $\hat{\mathbf{x}}_{k+1|k} \triangleq E\left(\mathbf{x}_{k+1}|\mathbf{Y}_{1:k}\right)$, and

$$\mathbf{F}_k \triangleq \frac{\partial \mathbf{f}_k\left(\hat{\mathbf{x}}_{k|k}, \mathbf{u}_k\right)}{\partial \mathbf{x}} \quad , \quad \mathbf{H}_{k+1} \triangleq \frac{d\mathbf{h}_{k+1}\left(\hat{\mathbf{x}}_{k+1|k}\right)}{d\mathbf{x}} \quad (5)$$

Discrete Extended Kalman Filter

Linearized Model:

Using (3)–(4), we can approximate system (1)–(2) by:

$$\mathbf{x}_{k+1} = \mathbf{F}_k \mathbf{x}_k + \mathbf{G}_k \mathbf{w}_k + \left(\mathbf{f}_k \left(\hat{\mathbf{x}}_{k|k}, \mathbf{u}_k \right) - \mathbf{F}_k \hat{\mathbf{x}}_{k|k} \right) \tag{6}$$

$$\mathbf{y}_{k+1} = \mathbf{H}_{k+1} \mathbf{x}_{k+1} + \mathbf{v}_{k+1} + \left(\mathbf{h}_{k+1} \left(\hat{\mathbf{x}}_{k+1|k} \right) - \mathbf{H}_{k+1} \hat{\mathbf{x}}_{k+1|k} \right) \tag{7}$$

Formulation Overview:

As follows, we obtain the discrete extended Kalman filter for system (1)–(2) as the discrete Kalman filter applied to (6)–(7).

Discrete Extended Kalman Filter

Discrete-Time Prediction:

Given the updated (or filtered) mean $\hat{\mathbf{x}}_{k|k}$ and covariance $\mathbf{P}_{k|k}$, we can calculate the predictive mean $\hat{\mathbf{x}}_{k+1|k}$ and covariance $\mathbf{P}_{k+1|k}$ by:

$$\hat{\mathbf{x}}_{k+1|k} = \mathbf{f}_k\left(\hat{\mathbf{x}}_{k|k}, \mathbf{u}_k\right) \tag{8}$$

$$\mathbf{P}_{k+1|k} = \mathbf{F}_k \mathbf{P}_{k|k} \mathbf{F}_k^{\mathrm{T}} + \mathbf{G}_k \mathbf{Q}_k \mathbf{G}_k^{\mathrm{T}} \tag{9}$$

Discrete Extended Kalman Filter

Other Predictive Expectations:

We can also show that

$$\hat{\mathbf{y}}_{k+1|k} = \mathbf{h}_{k+1}\left(\hat{\mathbf{x}}_{k+1|k}\right) \tag{10}$$

$$\mathbf{P}^{Y}_{k+1|k} = \mathbf{H}_{k+1}\mathbf{P}_{k+1|k}\mathbf{H}^{\mathrm{T}}_{k+1} + \mathbf{R}_{k+1} \tag{11}$$

$$\mathbf{P}^{XY}_{k+1|k} = \mathbf{P}_{k+1|k}\mathbf{H}^{\mathrm{T}}_{k+1} \tag{12}$$

Discrete Extended Kalman Filter

Update:

The updated (or filtered) mean and covariance are computed by

$$\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}\left(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k}\right) \tag{13}$$

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - \mathbf{P}^{XY}_{k+1|k}\left(\mathbf{P}^{Y}_{k+1|k}\right)^{-1}\left(\mathbf{P}^{XY}_{k+1|k}\right)^{\mathrm{T}} \tag{14}$$

where $\mathbf{K}_{k+1}$ is the Kalman gain, which is given by

$$\mathbf{K}_{k+1} = \mathbf{P}^{XY}_{k+1|k}\left(\mathbf{P}^{Y}_{k+1|k}\right)^{-1} \tag{15}$$

Discrete Extended Kalman Filter

Comments:

- The functions $\mathbf{f}_k$ and $\mathbf{h}_k$ must be differentiable w.r.t. $\mathbf{x}_k$.
- The EKF is a kind of benchmark in state estimation of non-linear systems subject to stochastic inputs.
- For the EKF to be stable and to show a good convergence rate, we must tune the matrices $\bar{\mathbf{P}}$ and $\mathbf{Q}_k$ (we are going to see that in the next computational exercise).

Algorithm: Discrete EKF

% Initialization:

$\hat{\mathtt{x}}_{1|1} \leftarrow \bar{\mathtt{x}} \qquad \mathtt{P}_{1|1} \leftarrow \bar{\mathtt{P}}$

Repeat for $k > 1$:

% Prediction:

$\hat{\mathtt{x}}_{k+1|k} \leftarrow \mathtt{f}_k(\hat{\mathtt{x}}_{k|k}, \mathtt{u}_k)$

$\mathtt{P}_{k+1|k} \leftarrow \mathtt{F}_k \mathtt{P}_{k|k} \mathtt{F}_k^{\mathrm{T}} + \mathtt{G}_k \mathtt{Q}_k \mathtt{G}_k^{\mathrm{T}}$

$\hat{\mathtt{y}}_{k+1|k} \leftarrow \mathtt{h}_{k+1}(\hat{\mathtt{x}}_{k+1|k})$

$\mathtt{P}^{Y}_{k+1|k} \leftarrow \mathtt{H}_{k+1} \mathtt{P}_{k+1|k} \mathtt{H}_{k+1}^{\mathrm{T}} + \mathtt{R}_{k+1}$

$\mathtt{P}^{XY}_{k+1|k} \leftarrow \mathtt{P}_{k+1|k} \mathtt{H}_{k+1}^{\mathrm{T}}$

% Update:

$\mathtt{K}_{k+1} \leftarrow \mathtt{P}^{XY}_{k+1|k} (\mathtt{P}^{Y}_{k+1|k})^{-1}$

$\hat{\mathtt{x}}_{k+1|k+1} \leftarrow \hat{\mathtt{x}}_{k+1|k} + \mathtt{K}_{k+1}(\mathtt{y}_{k+1} - \hat{\mathtt{y}}_{k+1|k})$

$\mathtt{P}_{k+1|k+1} \leftarrow \mathtt{P}_{k+1|k} - \mathtt{P}^{XY}_{k+1|k} (\mathtt{P}^{Y}_{k+1|k})^{-1} (\mathtt{P}^{XY}_{k+1|k})^{\mathrm{T}}$

end-repeat, $k \leftarrow k + 1$

References. . .

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