MP-208

Optimal Filtering with Aerospace Applications

Chapter 6: Extended Kalman Filter

Part II: Continuous-Discrete Formulation

Prof. Dr. Davi Antônio dos Santos
Instituto Tecnológico de Aeronáutica
www.professordavisantos.com

São José dos Campos - SP
2023

Contents

Problem Definition…

Problem Definition

State Equation:

Consider a continuous state SP $\{\mathbf{X}(t)\}$ and its realization $\{\mathbf{x}(t)\}$, with $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ dynamically described by

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right) + \mathbf{g}\left(\mathbf{x}(t)\right)\mathbf{w}(t) \tag{1}$$

where $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is a known input, $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is an unknown input, and $\mathbf{f} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and $\mathbf{g} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x \times n_w}$ are given non-linear functions.

Assume that:

1. The initial state $\mathbf{x}(t_0)$ is a realization of the approx. symmetric $\mathbf{X}(t_0) \sim (\bar{\mathbf{x}}, \bar{\mathbf{P}})$, where $\bar{\mathbf{x}} \in \mathbb{R}^{n_x}$ and $\bar{\mathbf{P}} \in \mathbb{R}^{n_x \times n_x}$ are known.
2. The signal $\{\mathbf{w}(t)\}$ is a realization of the uncorrelated SP $\{\mathbf{W}(t)\}$, with the approx. symmetric $\mathbf{W}(t) \sim (\mathbf{0}, \mathbf{Q}(t))$, where $\mathbf{Q}(t) \in \mathbb{R}^{n_w \times n_w}$ is known.
3. $\left\{\{\mathbf{W}(t)\}, \mathbf{X}(t_0)\right\}$ is mutually uncorrelated.

Problem Definition

Measurement Equation:

Consider a measurement SP $\{\mathbf{Y}_k\}$ and its realization $\{\mathbf{y}_k\}$, with $\mathbf{y}_{k+1} \in \mathbb{R}^{n_y}$ described by

$$\mathbf{y}_{k+1} = \mathbf{h}_{k+1}(\mathbf{x}_{k+1}) + \mathbf{v}_{k+1} \tag{2}$$

where $\mathbf{v}_{k+1} \in \mathbb{R}^{n_y}$ is an unknown input and $\mathbf{h}_{k+1} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ is a given non-linear function.

Assume that:

1. The sequence $\{\mathbf{v}_k\}$ is a realization of the uncorrelated SP $\{\mathbf{V}_k\}$, with the approx. symmetric $\mathbf{V}_k \sim (\mathbf{0}, \mathbf{R}_k)$, where $\mathbf{R}_k \in \mathbb{R}^{n_y \times n_y}$ is known.
2. $\{\{\mathbf{V}_k\}, \{\mathbf{W}_k\}, \mathbf{X}_1\}$ is mutually uncorrelated.

Problem Definition

Problem Statement:

The problem is to obtain an approximately optimal (MMSE) recursive filter for estimating $\left\{\mathbf{x}(t)\right\}$ using $\{\mathbf{y}_k\}$, $\left\{\mathbf{u}(t)\right\}$, and (1)–(2).

Comments:

In this course, we are going to present the following solutions to this problem:

i. EKF: Extended Kalman Filter (in this chapter);
ii. UKF: Unscented Kalman Filter; and
iii. EnKF: Ensemble Kalman Filter.

Continuous-Discrete Extended Kalman Filter. . .

Continuous-Discrete Extended Kalman Filter

Functional Approximations:

Let us approximate the non-linear functions $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}_k$ by Taylor series expansion and truncation:

$$\mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right) \approx \mathbf{f}\left(\hat{\mathbf{x}}(t), \mathbf{u}(t)\right) + \mathbf{F}(t)\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right) \tag{3}$$

$$\mathbf{g}\left(\mathbf{x}(t)\right) \approx \mathbf{g}\left(\hat{\mathbf{x}}(t)\right) \triangleq \mathbf{G}\left(t\right) \tag{4}$$

$$\mathbf{h}_{k+1}\left(\mathbf{x}_{k+1}\right) \approx \mathbf{h}_{k+1}\left(\hat{\mathbf{x}}_{k+1|k}\right) + \mathbf{H}_{k+1}\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right) \tag{5}$$

where $\hat{\mathbf{x}}(t) \triangleq E\left(\mathbf{x}(t)|\mathbf{Y}_{1:k}\right)$, $\hat{\mathbf{x}}_{k+1|k} \triangleq E\left(\mathbf{x}_{k+1}|\mathbf{Y}_{1:k}\right)$, and

$$\mathbf{F}(t) \triangleq \frac{\partial \mathbf{f}\left(\hat{\mathbf{x}}(t), \mathbf{u}(t)\right)}{\partial \mathbf{x}} \quad , \quad \mathbf{H}_{k+1} \triangleq \frac{d\mathbf{h}_{k+1}\left(\hat{\mathbf{x}}_{k+1|k}\right)}{d\mathbf{x}} \tag{6}$$

Continuous-Discrete Extended Kalman Filter

Linearized Model:

Using (3)–(5), we can approximate system (1)–(2) by:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{w}(t) + \Big(\mathbf{f}\left(\hat{\mathbf{x}}(t), \mathbf{u}(t)\right) - \mathbf{F}(t)\hat{\mathbf{x}}(t)\Big) \qquad (7)$$

$$\mathbf{y}_{k+1} = \mathbf{H}_{k+1}\mathbf{x}_{k+1} + \mathbf{v}_{k+1} + \left(\mathbf{h}_{k+1}\left(\hat{\mathbf{x}}_{k+1|k}\right) - \mathbf{H}_{k+1}\hat{\mathbf{x}}_{k+1|k}\right) \qquad (8)$$

Formulation Overview:

As follows, we obtain the continuous-discrete extended Kalman filter for system (1)–(2) as the continuous-discrete Kalman filter applied to (7)–(8).

Continuous-Discrete Extended Kalman Filter

Continuous-Time Prediction:

Given the filtered mean $\hat{\mathbf{x}}_{k|k}$ and covariance $\mathbf{P}_{k|k}$, we can calculate the predictive mean $\hat{\mathbf{x}}_{k+1|k} = \hat{\mathbf{x}}(t_{k+1})$ and covariance $\mathbf{P}_{k+1|k} = \mathbf{P}(t_{k+1})$, by integrating the ODEs:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}\left(\hat{\mathbf{x}}(t), \mathbf{u}(t)\right) \tag{9}$$

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}(t)^{\mathrm{T}} + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}(t)^{\mathrm{T}} \tag{10}$$

from t_k to t_{k+1}, with i.c. $\hat{\mathbf{x}}(t_k) = \hat{\mathbf{x}}_{k|k}$, $\mathbf{P}(t_k) = \mathbf{P}_{k|k}$.

Remarks:

- We consider that $\mathbf{u}(t) = \mathbf{u}_k, \forall t \in [t_k, t_{k+1})$.
- We usually adopt the 4th-order Runge-Kutta method to solve the ODEs (9)–(10).

Continuous-Discrete Extended Kalman Filter

Other Predictive Expectations:

We can also show that

$$\hat{\mathbf{y}}_{k+1|k} = \mathbf{h}_{k+1}\left(\hat{\mathbf{x}}_{k+1|k}\right) \tag{11}$$

$$\mathbf{P}^{Y}_{k+1|k} = \mathbf{H}_{k+1}\mathbf{P}_{k+1|k}\mathbf{H}^{\mathrm{T}}_{k+1} + \mathbf{R}_{k+1} \tag{12}$$

$$\mathbf{P}^{XY}_{k+1|k} = \mathbf{P}_{k+1|k}\mathbf{H}^{\mathrm{T}}_{k+1} \tag{13}$$

Continuous-Discrete Extended Kalman Filter

Update:

The updated (filtered) mean and covariance are computed by

$$\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}\left(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k}\right) \tag{14}$$

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - \mathbf{P}^{XY}_{k+1|k}\left(\mathbf{P}^{Y}_{k+1|k}\right)^{-1}\left(\mathbf{P}^{XY}_{k+1|k}\right)^{\mathrm{T}} \tag{15}$$

where $\mathbf{K}_{k+1}$ is the Kalman gain, which is given by

$$\mathbf{K}_{k+1} = \mathbf{P}^{XY}_{k+1|k}\left(\mathbf{P}^{Y}_{k+1|k}\right)^{-1} \tag{16}$$

Continuous-Discrete Extended Kalman Filter

Comments:

- The functions $\mathbf{f}$ and $\mathbf{h}_k$ must be differentiable w.r.t. $\mathbf{x}(t)$ and $\mathbf{x}_k$, respectively.
- The EKF is a kind of benchmark in state estimation of non-linear systems subject to stochastic inputs.
- For the EKF to be stable and to show a good convergence rate, we must tune the matrices $\bar{\mathbf{P}}$ and $\mathbf{Q}(t)$ (we are going to see that soon in a case study).

Algorithm: Continuous-Discrete EKF

% Initialization:

$\hat{x}_{1|1} \leftarrow \bar{x} \qquad P_{1|1} \leftarrow \bar{P}$

Repeat for $k > 1$:

% Prediction:

Integrate from t_k to t_{k+1}, with i.c. $\hat{x}_{k|k}$ and $P_{k|k}$:

$\dot{\hat{x}}(t) = f\left(\hat{x}(t), u_k\right)$

$\dot{P}(t) = F(t)P(t) + P(t)F(t)^{T} + G(t)Q(t)G(t)^{T}$

$\hat{x}_{k+1|k} \leftarrow \hat{x}(t_{k+1}) \qquad P_{k+1|k} \leftarrow P(t_{k+1})$

$\hat{y}_{k+1|k} \leftarrow h_{k+1}(\hat{x}_{k+1|k})$

$P^{Y}_{k+1|k} \leftarrow H_{k+1}P_{k+1|k}H^{T}_{k+1} + R_{k+1}$

$P^{XY}_{k+1|k} \leftarrow P_{k+1|k}H^{T}_{k+1}$

% Update:

$K_{k+1} \leftarrow P^{XY}_{k+1|k}(P^{Y}_{k+1|k})^{-1}$

$\hat{x}_{k+1|k+1} \leftarrow \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - \hat{y}_{k+1|k})$

$P_{k+1|k+1} \leftarrow P_{k+1|k} - P^{XY}_{k+1|k}(P^{Y}_{k+1|k})^{-1}(P^{XY}_{k+1|k})^{T}$

end-repeat, $k \leftarrow k + 1$

References. . .

References

Bar-Shalom, Y., Li, R. X., Kirubarajan, T. **Estimation with Applications to Tracking and Navigation**. New Jersey: Jonh Wiley & Sons, 2001.

Anderson, B. D. O., Moore, J. B. **Optimal Filtering**. New York: Dover, 2005.

Maybeck, P. S. **Stochastic Models, Estimation and Control**. New York: Academic Press, 1979.

Gelb, A. **Applied Optimal Estimation**. Cambridge: The M.I.T. Press, 1974.