MP-208

Optimal Filtering with Aerospace Applications

Chapter 7: Unscented Kalman Filter

Part II: Continuous-Discrete Formulation

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Problem Definition. . .

Problem Definition

State Equation:

Consider a continuous state SP $\left\{\mathbf{X}(t)\right\}$ and its realization $\left\{\mathbf{x}(t)\right\}$, with $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ dynamically described by

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right) + \mathbf{g}\left(\mathbf{x}(t)\right)\mathbf{w}(t) \tag{1}$$

where $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is a known input, $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is an unknown input, and $\mathbf{f} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and $\mathbf{g} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x \times n_w}$ are given non-linear functions.

Assume that:

1. The initial state $\mathbf{x}(t_0)$ is a realization of $\mathbf{X}(t_0) \sim (\bar{\mathbf{x}}, \bar{\mathbf{P}})$, where $\bar{\mathbf{x}} \in \mathbb{R}^{n_x}$ and $\bar{\mathbf{P}} \in \mathbb{R}^{n_x \times n_x}$ are known.
2. The signal $\left\{\mathbf{w}(t)\right\}$ is a realization of the uncorrelated SP $\left\{\mathbf{W}(t)\right\}$, with $\mathbf{W}(t) \sim (\mathbf{0}, \mathbf{Q}(t))$, where $\mathbf{Q}(t) \in \mathbb{R}^{n_w \times n_w}$ is known.
3. $\left\{\left\{\mathbf{W}(t)\right\}, \mathbf{X}(t_0)\right\}$ is uncorrelated.

Problem Definition

Measurement Equation:

Consider a measurement SP $\{\mathbf{Y}_k\}$ and its realization $\{\mathbf{y}_k\}$, with $\mathbf{y}_{k+1} \in \mathbb{R}^{n_y}$ described by

$$\mathbf{y}_{k+1} = \mathbf{h}_{k+1}(\mathbf{x}_{k+1}) + \mathbf{v}_{k+1} \tag{2}$$

where $\mathbf{v}_{k+1} \in \mathbb{R}^{n_y}$ is an unknown input and $\mathbf{h}_{k+1} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ is a given non-linear function.

Assume that:

1. The sequence $\{\mathbf{v}_k\}$ is a realization of the uncorrelated SP $\{\mathbf{V}_k\}$, with $\mathbf{V}_k \sim (\mathbf{0}, \mathbf{R}_k)$, where $\mathbf{R}_k \in \mathbb{R}^{n_y \times n_y}$ is known.
2. $\{\{\mathbf{V}_k\}, \{\mathbf{W}_k\}, \mathbf{X}_1\}$ is uncorrelated.

Problem Definition

Problem Statement:

The problem is to obtain an approximately optimal (MMSE) recursive filter to estimate $\{\mathbf{x}(t)\}$ using $\{\mathbf{y}_k\}$, $\{\mathbf{u}(t)\}$, and (1)–(2).

Comment:

In the previous section, we solved this problem using the CDEKF. Now, we formulate a different solution method: the continuous-discrete unscented Kalman filter (CDUKF). Let us start by defining the unscented integration (UI).

Unscented Integration…

Unscented Integration

Approximating the a Posteriori Distribution:

Consider the RVs $\mathbf{X}(t_1) : \Omega \to \mathbb{R}^{n_x}$ and $\mathbf{W} : \Omega \to \mathbb{R}^{n_w}$ as well as the differential equation:

$$\dot{\mathbf{X}}(t) = \mathbf{f}(\mathbf{X}(t), \mathbf{W}) \tag{3}$$

Assume that:

- $\mathbf{X}(t_1)$ has known mean and covariance $\bar{\mathbf{x}}(t_1)$, $\mathbf{P}(t_1)$.
- $\mathbf{W}$ has zero mean and known covariance $\mathbf{Q}$.
- $\mathbf{W}$ and $\mathbf{X}(t_1)$ are uncorrelated.

The mean $\bar{\mathbf{x}}(t_2)$ and the covariance $\mathbf{P}(t_2)$ of $\mathbf{X}(t_2)$ at $t_2 > t_1$ can be approximated by the unscented integration (UI).

Unscented Integration

Procedure (UI):

1. Obtain the σ-points ${}^a\mathcal{X}^i(t_1) \triangleq [\mathcal{X}^i(t_1)^{\mathrm{T}} \ (\mathcal{W}^i)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{n_a}$ of the augmented state ${}^a\mathbf{X}(t_1) \triangleq [\mathbf{X}(t_1)^{\mathrm{T}} \ \mathbf{W}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{n_a}$, with $n_a = n_x + n_w$, as well as the respective weights:

$$\left\{ \left({}^a\mathcal{X}^i(t_1), \rho^i \right), i = 0, ..., 2n_a \right\} \leftarrow \mathrm{SP}\left({}^a\bar{\mathbf{x}}(t_1), {}^a\mathbf{P}(t_1) \right) \tag{4}$$

   where

$${}^a\bar{\mathbf{x}}(t_1) = \begin{bmatrix} \bar{\mathbf{x}}(t_1) \\ \mathbf{0}_{n_w \times 1} \end{bmatrix} \quad , \quad {}^a\mathbf{P}(t_1) = \begin{bmatrix} \mathbf{P}(t_1) & \mathbf{0}_{n_x \times n_w} \\ \mathbf{0}_{n_w \times n_x} & \mathbf{Q} \end{bmatrix} \tag{5}$$

2. Integrate the following ODEs from t_1 to t_2:

$$\dot{\mathcal{X}}^i(t) = \mathbf{f}\left(\mathcal{X}^i(t), \mathcal{W}^i \right), \tag{6}$$

   with initial conditions $\mathcal{X}^i(t_1)$, to obtain $\mathcal{X}^i(t_2)$, for $i = 0, ..., 2n_a$.

Unscented Integration

3. Approximate the *a posteriori* mean and covariance by sample statistics using the transformed σ-points $\mathcal{X}^i(t_2)$, *i.e.*:

$$\bar{\mathbf{x}}(t_2) \approx \sum_{i=0}^{2n_a} \rho^i \mathcal{X}^i(t_2) \tag{7}$$

$$\mathbf{P}(t_2) \approx \sum_{i=0}^{2n_a} \rho^i \left(\mathcal{X}^i(t_2) - \bar{\mathbf{x}}(t_2)\right) \left(\mathcal{X}^i(t_2) - \bar{\mathbf{x}}(t_2)\right)^{\mathrm{T}} \tag{8}$$

Unscented Integration

Comments:

1. Note that the set $\{{}^a\mathcal{X}^i(t_1), i = 0, ..., 2n_a\}$ is a deterministic sample of ${}^a\mathbf{X}(t_1)$.
2. We know that, alternatively, the *a posteriori* mean and covariance can be approximated by first linearizing $\mathbf{f}$ and then using the linearity property of $E(.)$.
3. In general, the UI approximation is as good as the one obtained by the 2nd-order functional approximation of $\mathbf{f}$. In particular, if the *a priori* RV ${}^a\mathbf{X}(t_1)$ is Gaussian, the UI is 3rd-order accurate.
4. A procedure that is equivalent (one must verify!) to the afore-described UI is presented in (Sarkka, 2007).

Continuous-Discrete Unscented Kalman Filter. . .

Continuous-Discrete Unscented Kalman Filter

Formulation Overview:

The continuous-discrete unscented Kalman filter (CDUKF) has the same structure as the CDEKF. The only difference between them is that the former approximates the predictive expected values of the prediction phase by using both UT and UI instead of Taylor-series linearization.

Continuous-Discrete Unscented Kalman Filter

Obtain the σ-points:

Consider the system (1)–(2) and define the augmented state RV

$$ {}^{a}\mathbf{X}(t_k) \triangleq \begin{bmatrix} \mathbf{X}(t_k) \\ \mathbf{W}(t_k) \\ \mathbf{V}_{k+1} \end{bmatrix} \in \mathbb{R}^{n_a} \tag{9} $$

where $n_a \triangleq n_x + n_w + n_y$.

From the problem definition and using the notation so far adopted for filtered mean and covariance, we obtain the mean and covariance of ${}^{a}\mathbf{X}(t_k)$ as

$$ {}^{a}\bar{\mathbf{x}}(t_k) \triangleq \begin{bmatrix} \hat{\mathbf{x}}_{k|k} \\ \mathbf{0}_{n_w \times 1} \\ \mathbf{0}_{n_y \times 1} \end{bmatrix} \quad , \quad {}^{a}\mathbf{P}(t_k) \triangleq \begin{bmatrix} \mathbf{P}_{k|k} & \mathbf{0}_{n_x \times n_w} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_w \times n_x} & \mathbf{Q}(t_k) & \mathbf{0}_{n_w \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_w} & \mathbf{R}_{k+1} \end{bmatrix} \tag{10} $$

Continuous-Discrete Unscented Kalman Filter

Obtain the σ-points *(cont.)*:

The σ-points ${}^{a}\mathcal{X}^{i}(t_k) \in \mathbb{R}^{n_a}$ of ${}^{a}\mathbf{X}(t_k)$ are given by

$$\left\{ \left({}^{a}\mathcal{X}^{i}(t_k), \rho^{i} \right), i = 0, ..., 2n_a \right\} \leftarrow \mathrm{SP}\left({}^{a}\bar{\mathbf{x}}(t_k), {}^{a}\mathbf{P}(t_k) \right) \tag{11}$$

and can be partitioned as

$${}^{a}\mathcal{X}^{i}(t_k) \triangleq \begin{bmatrix} \mathcal{X}^{i}(t_k) \\ \mathcal{W}^{i}(t_k) \\ \mathcal{V}^{i}_{k+1} \end{bmatrix} \tag{12}$$

where $\mathcal{X}^{i}(t_k) \in \mathbb{R}^{n_x}$, $\mathcal{W}^{i}(t_k) \in \mathbb{R}^{n_w}$, and $\mathcal{V}^{i}_{k+1} \in \mathbb{R}^{n_y}$ are σ-points of $\mathbf{X}(t_k)$, $\mathbf{W}(t_k)$, and $\mathbf{V}_{k+1}$, respectively.

Continuous-Discrete Unscented Kalman Filter

Transforming the σ-points:

Consider the σ-points $\mathcal{X}^i(t_k)$ as initial conditions and integrate the following ODEs from t_k to t_{k+1}:

$$\dot{\mathcal{X}}^i(t) = \mathbf{f}\left(\mathcal{X}^i(t), \mathbf{u}(t_k)\right) + \mathbf{g}\left(\mathcal{X}^i(t)\right)\mathcal{W}^i(t_k), \tag{13}$$

to obtain the σ-points $\mathcal{X}^i_{k+1|k} \triangleq \mathcal{X}^i(t_{k+1})$ of the predictive state, for $i = 0, ..., 2n_a$. In this integration, consider that $\mathbf{u}(t)$ and $\mathcal{W}^i(t)$ keep constant during the time interval $t \in [t_k, t_{k+1}]$.

The σ-points $\mathcal{X}^i_{k+1|k}$ and $\mathcal{V}^i_{k+1}$, when transformed by (2), give rise to the σ-points of the predictive measure:

$$\mathcal{Y}^i_{k+1|k} = \mathbf{h}_{k+1}\left(\mathcal{X}^i_{k+1|k}\right) + \mathcal{V}^i_{k+1} \tag{14}$$

for $i = 0, ..., 2n_a$.

Continuous-Discrete Unscented Kalman Filter

Continuous-Time Prediction:

The predictive expected values are then immediately approximated by sample statistics on the predictive σ-points, *i.e.*:

$$\hat{\mathbf{x}}_{k+1|k} \approx \sum_{i=0}^{2n_a} \rho^i \; \mathcal{X}^i_{k+1|k} \tag{15}$$

$$\mathbf{P}_{k+1|k} \approx \sum_{i=0}^{2n_a} \rho^i \; \left(\mathcal{X}^i_{k+1|k} - \hat{\mathbf{x}}_{k+1|k}\right)\left(\mathcal{X}^i_{k+1|k} - \hat{\mathbf{x}}_{k+1|k}\right)^{\mathrm{T}} \tag{16}$$

$$\hat{\mathbf{y}}_{k+1|k} \approx \sum_{i=0}^{2n_a} \rho^i \; \mathcal{Y}^i_{k+1|k} \tag{17}$$

...

Continuous-Discrete Unscented Kalman Filter

...

$$\mathbf{P}^{y}_{k+1|k} \approx \sum_{i=0}^{2n_a} \rho^i \left(\mathcal{Y}^i_{k+1|k} - \hat{\mathbf{y}}_{k+1|k}\right)\left(\mathcal{Y}^i_{k+1|k} - \hat{\mathbf{y}}_{k+1|k}\right)^{\mathrm{T}} \tag{18}$$

$$\mathbf{P}^{xy}_{k+1|k} \approx \sum_{i=0}^{2n_a} \rho^i \left(\mathcal{X}^i_{k+1|k} - \hat{\mathbf{x}}_{k+1|k}\right)\left(\mathcal{Y}^i_{k+1|k} - \hat{\mathbf{y}}_{k+1|k}\right)^{\mathrm{T}} \tag{19}$$

Continuous-Discrete Unscented Kalman Filter

Update:

The update step of the CDUKF is carried out as in the DUKF, *i.e.*, by computing

$$\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}\left(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k}\right) \tag{20}$$

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - \mathbf{P}^{xy}_{k+1|k}\left(\mathbf{P}^{y}_{k+1|k}\right)^{-1}\left(\mathbf{P}^{xy}_{k+1|k}\right)^{\mathrm{T}} \tag{21}$$

where $\mathbf{K}_{k+1}$ is the Kalman gain, which is given by

$$\mathbf{K}_{k+1} = \mathbf{P}^{xy}_{k+1|k}\left(\mathbf{P}^{y}_{k+1|k}\right)^{-1} \tag{22}$$

Continuous-Discrete Unscented Kalman Filter

Comments:

- We expect a considerably higher computational burden of the CDUKF compared to the DUKF, since at each iteration, it is necessary to numerically integrate $2n_a + 1$ ODEs.
- The continuous-discrete formulation has the following advantages: 1) it does not require the time discretization of the time-continuous model; 2) it allows a simpler implementation, since the continuous-time model is generally more compact; and 3) it is attractive for augmented adaptive filters, since the parameters of the continuous-time model have physical meaning.

References. . .

References

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