MP-208
Optimal Filtering with Aerospace Applications
Chapter 8: Ensemble Kalman Filter
Part I: Discrete-Time Formulation

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Contents

Problem Definition. . .

Problem Definition

State Equation:

Consider a state SP $\{\mathbf{X}_k\}$ and its realization $\{\mathbf{x}_k\}$, with $\mathbf{x}_k \in \mathbb{R}^{n_x}$ dynamically described by

$$\mathbf{x}_{k+1} = \mathbf{f}_k\left(\mathbf{x}_k, \mathbf{u}_k\right) + \mathbf{G}_k \mathbf{w}_k \qquad (1)$$

where $\mathbf{u}_k \in \mathbb{R}^{n_u}$ is a known input, $\mathbf{w}_k \in \mathbb{R}^{n_w}$ is an unknown input, $\mathbf{f}_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ is a given non-linear function, and $\mathbf{G}_k \in \mathbb{R}^{n_x \times n_w}$ is a known matrix.

Assume that:

1. The initial state $\mathbf{x}_1$ is a realization of $\mathbf{X}_1$, which is assumed to be Gaussian and to have known mean $\bar{\mathbf{x}} \in \mathbb{R}^{n_x}$ and covariance $\bar{\mathbf{P}} \in \mathbb{R}^{n_x \times n_x}$. For short, we denote $\mathbf{X}_1 \sim \mathcal{N}(\bar{\mathbf{x}}, \bar{\mathbf{P}})$.
2. The sequence $\{\mathbf{w}_k\}$ is a realization of an uncorrelated SP $\{\mathbf{W}_k\}$, with $\mathbf{W}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)$, where $\mathbf{Q}_k \in \mathbb{R}^{n_w \times n_w}$ is known.
3. $\left\{\{\mathbf{W}_k\}, \mathbf{X}_1\right\}$ is uncorrelated.

Problem Definition

Measurement Equation:

Consider a measurement SP $\{\mathbf{Y}_k\}$ and its realization $\{\mathbf{y}_k\}$, with $\mathbf{y}_{k+1} \in \mathbb{R}^{n_y}$ described by

$$\mathbf{y}_{k+1} = \mathbf{h}_{k+1}(\mathbf{x}_{k+1}) + \mathbf{v}_{k+1} \tag{2}$$

where $\mathbf{v}_{k+1} \in \mathbb{R}^{n_y}$ is an unknown input and $\mathbf{h}_{k+1} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ is a given non-linear function.

Assume that:

1. The sequence $\{\mathbf{v}_k\}$ is a realization of the uncorrelated SP $\{\mathbf{V}_k\}$, with $\mathbf{V}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$, where $\mathbf{R}_k \in \mathbb{R}^{n_y \times n_y}$ is known.
2. $\left\{\{\mathbf{V}_k\}, \{\mathbf{W}_k\}, \mathbf{X}_1\right\}$ is uncorrelated.

Problem Definition

Problem Statement:

The problem is to obtain an approximately optimal (MMSE) recursive filter for estimating $\{\mathbf{x}_k\}$ using $\{\mathbf{y}_k\}$, $\{\mathbf{u}_k\}$, and (1)–(2).

Comment:

In the previous sections, we solved this problem using the EKF and the UKF. Now, we formulate a different solution method: the ensemble Kalman filter (EnKF).

Discrete Ensemble Kalman Filter. . .

Discrete Ensemble Kalman Filter

Formulation Overview:

The Discrete Ensemble Kalman Filter (DEnKF) has the same structure as the DUKF. The only difference between them is that the former approximates the predictive expected values of the prediction phase by using statistics over a random sample (rather than over the σ-points) and, instead of updating the mean and covariance, it updates the particles.

Discrete Ensemble Kalman Filter

Random Sampling:

Respecting the statistical models established in the problem definition, we can draw the following samples from the stochastic inputs of the system:

- Initial state $\mathbf{X}_1$:

$$\mathcal{X}_{1|1}^i \sim \mathcal{N}(\bar{\mathbf{x}}, \bar{\mathbf{P}}), \quad i = 1, \ldots, N \tag{3}$$

- State noise $\mathbf{W}_k$:

$$\mathcal{W}_k^i \sim \mathcal{N}(\mathbf{0}_{n_w \times 1}, \mathbf{Q}_k), \quad i = 1, \ldots, N \tag{4}$$

- Measurement noise $\mathbf{V}_{k+1}$:

$$\mathcal{V}_{k+1}^i \sim \mathcal{N}(\mathbf{0}_{n_y \times 1}, \mathbf{R}_{k+1}), \quad i = 1, \ldots, N \tag{5}$$

Discrete Ensemble Kalman Filter

Sample Transform:

By transforming the points $\mathcal{X}_{k|k}^i$ e $\mathcal{W}_k^i$ through (1), we obtain the predictive state sample $\mathcal{X}_{k+1|k}^i \in \mathbb{R}^{n_x}, i = 1, \ldots, N$, with

$$\mathcal{X}_{k+1|k}^i = \mathbf{f}_k \left(\mathcal{X}_{k|k}^i, \mathbf{u}_k \right) + \mathbf{G}_k \mathcal{W}_k^i \tag{6}$$

On the other hand, by transforming the points $\mathcal{X}_{k+1|k}^i$ and $\mathcal{V}_{k+1}^i$ through the measurement equation (2), we obtain the predictive measurement sample $\mathcal{Y}_{k+1|k}^i \in \mathbb{R}^{n_y}, i = 1, \ldots, N$:

$$\mathcal{Y}_{k+1|k}^i = \mathbf{h}_{k+1} \left(\mathcal{X}_{k+1|k}^i \right) + \mathcal{V}_{k+1}^i \tag{7}$$

Discrete Ensemble Kalman Filter

Discrete-time Prediction:

Therefore, the predictive expected values can be estimated using sampling statistics over $\mathcal{Y}^i_{k+1|k}$ and $\mathcal{X}^i_{k+1|k}$, for $i = 1, \ldots, N$, *i.e.*,

$$\hat{\mathbf{x}}_{k+1|k} \approx \frac{1}{N} \sum_{i=1}^{N} \mathcal{X}^i_{k+1|k} \tag{8}$$

$$\mathbf{P}_{k+1|k} \approx \frac{1}{N-1} \sum_{i=1}^{N} \left(\mathcal{X}^i_{k+1|k} - \hat{\mathbf{x}}_{k+1|k} \right) \left(\mathcal{X}^i_{k+1|k} - \hat{\mathbf{x}}_{k+1|k} \right)^{\mathrm{T}} \tag{9}$$

$$\hat{\mathbf{y}}_{k+1|k} \approx \frac{1}{N} \sum_{i=1}^{N} \mathcal{Y}^i_{k+1|k} \tag{10}$$

⋮

Discrete Ensemble Kalman Filter

(cont.)

$$\mathbf{P}^{y}_{k+1|k} \approx \frac{1}{N-1} \sum_{i=1}^{N} \left(\mathcal{Y}^{i}_{k+1|k} - \hat{\mathbf{y}}_{k+1|k} \right) \left(\mathcal{Y}^{i}_{k+1|k} - \hat{\mathbf{y}}_{k+1|k} \right)^{\mathrm{T}} \tag{11}$$

$$\mathbf{P}^{xy}_{k+1|k} \approx \frac{1}{N-1} \sum_{i=1}^{N} \left(\mathcal{X}^{i}_{k+1|k} - \hat{\mathbf{x}}_{k+1|k} \right) \left(\mathcal{Y}^{i}_{k+1|k} - \hat{\mathbf{y}}_{k+1|k} \right)^{\mathrm{T}} \tag{12}$$

Discrete Ensemble Kalman Filter

Updating/Filtering:

Now, we simply use the classical update equations of the Kalman filter to compute the filtered state sample:

$$\mathcal{X}^i_{k+1|k+1} = \mathcal{X}^i_{k+1|k} + \mathbf{K}_{k+1}\left(\mathbf{y}_{k+1} - \mathcal{Y}^i_{k+1|k}\right) \tag{13}$$

for $i = 1, ..., N$, where $\mathbf{K}_{k+1}$ is the Kalman gain, given by

$$\mathbf{K}_{k+1} = \mathbf{P}^{xy}_{k+1|k}\left(\mathbf{P}^{y}_{k+1|k}\right)^{-1} \tag{14}$$

Discrete Ensemble Kalman Filter

...Updating/Filtering:

Finally, by using sampling statistics, we can immediately approximate the filtered mean and covariance:

$$\hat{\mathbf{x}}_{k+1|k+1} \approx \frac{1}{N} \sum_{i=1}^{N} \mathcal{X}^i_{k+1|k+1} \tag{15}$$

$$\mathbf{P}_{k+1|k+1} \approx \frac{1}{N-1} \sum_{i=1}^{N} \left(\mathcal{X}^i_{k+1|k+1} - \hat{\mathbf{x}}_{k+1|k+1} \right) \left(\mathcal{X}^i_{k+1|k+1} - \hat{\mathbf{x}}_{k+1|k+1} \right)^{\mathrm{T}} \tag{16}$$

Discrete Ensemble Kalman Filter

Comments:

- The EnKF belongs to the class of particle filters (Daum and Huang, 2003; Kotecha and Djuric, 2003).
- The EnKF is widely explored in the weather forcasting literature. This area deals with nonlinear models of high order, with very uncertain initial estimates, and, commonly uses a large number of sensors (Daley,1991; Kalnay, 2003; Evensen, 1997).
- As reported in the literature, a small number N of samples (particles) is sufficient for the EnKF to show a good performance.

References. . .

References

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