MP-208

Optimal Filtering with Aerospace Applications

Chapter 8: Ensemble Kalman Filter

Part II: Continous-Discrete Formulation

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Problem Definition

State Equation:

Consider a continuous state SP $\{\mathbf{X}(t)\}$ and its realization $\{\mathbf{x}(t)\}$, with $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ dynamically described by

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right) + \mathbf{g}\left(\mathbf{x}(t)\right)\mathbf{w}(t) \tag{1}$$

where $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is a known input, $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is an unknown input, and $\mathbf{f} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and $\mathbf{g} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x \times n_w}$ are given non-linear functions.

Assume that:

1. The initial state $\mathbf{x}(t_0)$ is a realization of $\mathbf{X}(t_0) \sim \mathcal{N}(\bar{\mathbf{x}}, \bar{\mathbf{P}})$, where $\bar{\mathbf{x}} \in \mathbb{R}^{n_x}$ and $\bar{\mathbf{P}} \in \mathbb{R}^{n_x \times n_x}$ are known.
2. The signal $\{\mathbf{w}(t)\}$ is a realization of the uncorrelated SP $\{\mathbf{W}(t)\}$, with $\mathbf{W}(t) \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}(t))$, where $\mathbf{Q}(t) \in \mathbb{R}^{n_w \times n_w}$ is known.
3. $\left\{\{\mathbf{W}(t)\}, \mathbf{X}(t_0)\right\}$ is uncorrelated.

Problem Definition

Measurement Equation:

Consider a measurement SP $\{\mathbf{Y}_k\}$ and its realization $\{\mathbf{y}_k\}$, with $\mathbf{y}_{k+1} \in \mathbb{R}^{n_y}$ described by

$$\mathbf{y}_{k+1} = \mathbf{h}_{k+1}(\mathbf{x}_{k+1}) + \mathbf{v}_{k+1} \tag{2}$$

where $\mathbf{v}_{k+1} \in \mathbb{R}^{n_y}$ is an unknown input and $\mathbf{h}_{k+1} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ is a given non-linear function.

Assume that:

1. The sequence $\{\mathbf{v}_k\}$ is a realization of the uncorrelated SP $\{\mathbf{V}_k\}$, with $\mathbf{V}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$, where $\mathbf{R}_k \in \mathbb{R}^{n_y \times n_y}$ is known.
2. $\{\{\mathbf{V}_k\}, \{\mathbf{W}_k\}, \mathbf{X}_1\}$ is uncorrelated.

Problem Definition

Problem Statement:

The problem is to obtain an approximately optimal (MMSE) recursive filter to estimate $\{\mathbf{x}(t)\}$ using $\{\mathbf{y}_k\}$, $\{\mathbf{u}(t)\}$, and (1)–(2).

Comment:

In the previous sections, we solved this problem using the CDEKF and the CDUKF. Now, we formulate a different solution method: the continuous-discrete ensemble Kalman filter (CDEnKF).

Continuous-Discrete Ensemble Kalman Filter...

Continuous-Discrete Ensemble Kalman Filter

Formulation Overview:

The Continuous-Discrete Ensemble Kalman Filter (CDEnKF) has the same structure as the CDUKF. The only difference between them is that the former approximates the predictive expected values of the prediction phase by using statistics over a random sample (rather than over the σ-points) and, instead of updating the mean and covariance, it updates the particles.

Continuous-Discrete Ensemble Kalman Filter

Random Sampling:

Respecting the statistical models established in the problem definition, we can draw the following samples from the stochastic inputs of the system:

- Initial state $\mathbf{X}_1$:

$$\mathcal{X}_{1|1}^{i} \sim \mathcal{N}(\bar{\mathbf{x}}, \bar{\mathbf{P}}), \quad i = 1, \ldots, N \tag{3}$$

- State noise $\mathbf{W}(t_k)$:

$$\mathcal{W}^{i}(t_k) \sim \mathcal{N}(\mathbf{0}_{n_w \times 1}, \mathbf{Q}(t_k)), \quad i = 1, \ldots, N \tag{4}$$

- Measurement noise $\mathbf{V}_{k+1}$:

$$\mathcal{V}_{k+1}^{i} \sim \mathcal{N}(\mathbf{0}_{n_y \times 1}, \mathbf{R}_{k+1}), \quad i = 1, \ldots, N \tag{5}$$

Continuous-Discrete Ensemble Kalman Filter

Sample Transform:

Considering the initial conditions $\mathcal{X}^i(t_k), i = 1, \ldots, N,$ and integrating the following ODEs in $[t_k, t_{k+1}]$:

$$\dot{\mathcal{X}}^i(t) = \mathbf{f}\left(\mathcal{X}^i(t), \mathbf{u}(t_k)\right) + \mathbf{g}\left(\mathcal{X}^i(t)\right)\mathcal{W}^i(t_k) \tag{6}$$

we obtain the predictive state sample $\mathcal{X}^i_{k+1|k} \triangleq \mathcal{X}^i(t_{k+1}),\ i = 1, ..., N$. For that, we assume $\mathbf{u}(t)$ and $\mathcal{W}^i(t)$ keep constant in $t \in [t_k, t_{k+1})$.

On the other hand, by transforming the sample points $\mathcal{X}^i_{k+1|k}$ and $\mathcal{V}^i_{k+1}$ through the measurement equation (2), we obtain the predictive measurement sample $\mathcal{Y}^i_{k+1|k} \in \mathbb{R}^{n_y}, i = 1, \ldots, N$:

$$\mathcal{Y}^i_{k+1|k} = \mathbf{h}_{k+1}\left(\mathcal{X}^i_{k+1|k}\right) + \mathcal{V}^i_{k+1} \tag{7}$$

Continuous-Discrete Ensemble Kalman Filter

Continuous-Discrete Prediction:

Therefore, the predictive expected values can be estimated using sampling statistics over $\mathcal{Y}^i_{k+1|k}$ and $\mathcal{X}^i_{k+1|k}$, for $i = 1, \ldots, N$, *i.e.*,

$$\hat{\mathbf{x}}_{k+1|k} \approx \frac{1}{N} \sum_{i=1}^{N} \mathcal{X}^i_{k+1|k} \tag{8}$$

$$\mathbf{P}_{k+1|k} \approx \frac{1}{N-1} \sum_{i=1}^{N} \left(\mathcal{X}^i_{k+1|k} - \hat{\mathbf{x}}_{k+1|k} \right) \left(\mathcal{X}^i_{k+1|k} - \hat{\mathbf{x}}_{k+1|k} \right)^{\mathrm{T}} \tag{9}$$

$$\hat{\mathbf{y}}_{k+1|k} \approx \frac{1}{N} \sum_{i=1}^{N} \mathcal{Y}^i_{k+1|k} \tag{10}$$

⋮

Continuous-Discrete Ensemble Kalman Filter

(*cont.*)

$$\mathbf{P}^{y}_{k+1|k} \approx \frac{1}{N-1} \sum_{i=1}^{N} \left(\mathcal{Y}^{i}_{k+1|k} - \hat{\mathbf{y}}_{k+1|k} \right) \left(\mathcal{Y}^{i}_{k+1|k} - \hat{\mathbf{y}}_{k+1|k} \right)^{\mathrm{T}} \qquad (11)$$

$$\mathbf{P}^{xy}_{k+1|k} \approx \frac{1}{N-1} \sum_{i=1}^{N} \left(\mathcal{X}^{i}_{k+1|k} - \hat{\mathbf{x}}_{k+1|k} \right) \left(\mathcal{Y}^{i}_{k+1|k} - \hat{\mathbf{y}}_{k+1|k} \right)^{\mathrm{T}} \qquad (12)$$

Continuous-Discrete Ensemble Kalman Filter

Updating/Filtering:

Now, just as in the DEnKF, we simply use the classical update equations of the Kalman filter to compute the filtered state sample:

$$\mathcal{X}^i_{k+1|k+1} = \mathcal{X}^i_{k+1|k} + \mathbf{K}_{k+1}\left(\mathbf{y}_{k+1} - \mathcal{Y}^i_{k+1|k}\right) \tag{13}$$

for $i = 1, ..., N$, where $\mathbf{K}_{k+1}$ is the Kalman gain, given by

$$\mathbf{K}_{k+1} = \mathbf{P}^{xy}_{k+1|k}\left(\mathbf{P}^{y}_{k+1|k}\right)^{-1} \tag{14}$$

Continuous-Discrete Ensemble Kalman Filter

...Updating/Filtering:

Finally, by using sampling statistics, we can immediately approximate the filtered mean and covariance:

$$\hat{\mathbf{x}}_{k+1|k+1} \approx \frac{1}{N} \sum_{i=1}^{N} \mathcal{X}^{i}_{k+1|k+1} \tag{15}$$

$$\mathbf{P}_{k+1|k+1} \approx \frac{1}{N-1} \sum_{i=1}^{N} \left(\mathcal{X}^{i}_{k+1|k+1} - \hat{\mathbf{x}}_{k+1|k+1} \right) \left(\mathcal{X}^{i}_{k+1|k+1} - \hat{\mathbf{x}}_{k+1|k+1} \right)^{\mathrm{T}} \tag{16}$$

Continuous-Discrete Ensemble Kalman Filter

Comments:

- One can think that in the filtering phase we could use the mean and covariance updating equations of the classical Kalman filter instead of updating the random sample (particles) by (13). However, if we did that, at each iteration of the algorithm, we would need to sample the respective filtered pdf, which is rather unknown.

References. . .

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