MP-208
Optimal Filtering with Aerospace Applications
Chapter 9: Attitude Determination for Drones

Prof. Dr. Davi Antônio dos Santos
Instituto Tecnológico de Aeronáutica
www.professordavisantos.com

São José dos Campos - SP
2023

Contents

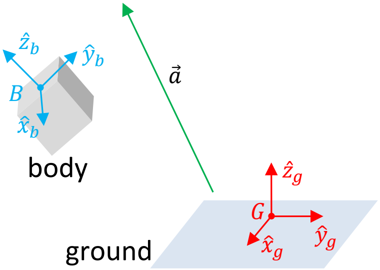
Introduction...

Scope of the Chapter

- This chapter is concerned with a three-dimensional attitude determination scheme for drones.
- The adopted attitude/navigation sensors are rate gyros, accelerometers, and magnetometers, all three-axial.
- This presentation only formulates the problem. The problem solution will be presented by the students, using CDEKF, CDUKF, and CDEnKF as the final evaluation of the course, according to a specific assignment.

Notation

- $\vec{a}$: geometric (or physical) vector.
- $\hat{a}$: geometric (or physical) vector.
- B, G: tridimensional points.
- $\mathcal{S}_b \triangleq \{B; \hat{x}_b, \hat{y}_b, \hat{z}_b\}$: body Cartesian coordinate system (CCS).
- $\mathcal{S}_g \triangleq \{G; \hat{x}_g, \hat{y}_g, \hat{z}_g\}$: ground CCS.



Notation (*cont.*)

- $\mathbf{a}_b \in \mathbb{R}^3$: representation of $\vec{a}$ in $\mathcal{S}_b$ (algebraic vector).
- $\mathbf{a}_g \in \mathbb{R}^3$: representation of $\vec{a}$ in $\mathcal{S}_g$ (algebraic vector).
- $\mathbf{D}^{b/g} \in \mathrm{SO}(3)$: attitude matrix of $\mathcal{S}_b$ w.r.t. $\mathcal{S}_g$. [1]

We can convert representations of a given geometric vector as follows:

$$\mathbf{a}_b = \mathbf{D}^{b/g}\mathbf{a}_g$$

From this and the definition of $\mathrm{SO}(3)$, we see that

$$\left(\mathbf{D}^{b/g}\right)^{-1} = \left(\mathbf{D}^{b/g}\right)^{\mathrm{T}} = \mathbf{D}^{g/b}$$

[1] $\mathrm{SO}(3) \triangleq \{\mathbf{D} \in \mathbb{R}^{3\times 3} : \mathbf{D}\mathbf{D}^{\mathrm{T}} = \mathbf{I}_3\}$ is the special orthogonal group.

Notation (*cont.*)

Consider the elementary rotation matrices (about axes 1, 2, and 3, resp.):

$$\mathbf{D}_1(\varrho) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\varrho & \mathrm{s}\varrho \\ 0 & -\mathrm{s}\varrho & \mathrm{c}\varrho \end{bmatrix} \quad \mathbf{D}_2(\varrho) = \begin{bmatrix} \mathrm{c}\varrho & 0 & -\mathrm{s}\varrho \\ 0 & 1 & 0 \\ \mathrm{s}\varrho & 0 & \mathrm{c}\varrho \end{bmatrix}$$

$$\mathbf{D}_3(\varrho) = \begin{bmatrix} \mathrm{c}\varrho & \mathrm{s}\varrho & 0 \\ -\mathrm{s}\varrho & \mathrm{c}\varrho & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For example, considering a 1-2-3 sequence of rotations of angles denoted by ϕ, θ, and ψ, respectively, the relationship between these (Euler) angles and the attitude matrix is

$$\mathbf{D}^{b/g} = \mathbf{D}_3(\psi)\mathbf{D}_2(\theta)\mathbf{D}_1(\phi)$$

Kinematic equations. . .

Kinematic Equations

Attitude Kinematics in Euler Angles 1-2-3

We can show that the attitude kinematics can be described in Euler angles in the 1-2-3 sequence by

$$\dot{\boldsymbol{\alpha}}^{b/g} = \mathbf{A}\left(\boldsymbol{\alpha}^{b/g}\right)\boldsymbol{\omega}_b^{b/g} \quad (1)$$

where $\boldsymbol{\alpha}^{b/g} \triangleq (\phi, \theta, \psi)$, $\boldsymbol{\omega}_b^{b/g} \in \mathbb{R}^3$ is the $\mathcal{S}_b$ representation of the angular velocity of $\mathcal{S}_b$ w.r.t. $\mathcal{S}_g$, and

$$\mathbf{A}\left(\boldsymbol{\alpha}^{b/g}\right) \triangleq \begin{bmatrix} \mathrm{c}\psi/\mathrm{c}\theta & -\mathrm{s}\psi/\mathrm{c}\theta & 0 \\ \mathrm{s}\psi & \mathrm{c}\psi & 0 \\ -\mathrm{c}\psi\mathrm{s}\theta/\mathrm{c}\theta & \mathrm{s}\psi\mathrm{s}\theta/\mathrm{c}\theta & 1 \end{bmatrix}$$

Sensor modeling. . .

Sensor Modeling

Rate Gyro

Its measure $\check{\boldsymbol{\omega}}_b \in \mathbb{R}^3$ can be modeled by

$$\check{\boldsymbol{\omega}}_b = \boldsymbol{\omega}_b^{b/g} + \boldsymbol{\beta}_b^{gy} + \mathbf{w}_b^{gy} \tag{2}$$

where $\mathbf{w}_b^{gy} \in \mathbb{R}^3$ is a zero-mean white Gaussian noise with covariance $\mathbf{Q}^{gy}$, which, for simplicity, is assumed to be constant and known, and $\boldsymbol{\beta}_b^{gy} \in \mathbb{R}^3$ is a drifting bias described by the following Wiener process:

$$\dot{\boldsymbol{\beta}}_b^{gy} = \mathbf{w}_b^{d,gy} \tag{3}$$

where $\mathbf{w}_b^{d,gy}$ is assumed to be a zero-mean white Gaussian noise with known covariance $\mathbf{Q}^{d,gy}$.

Accelerometer

Its measure $\check{\mathbf{a}}_b \in \mathbb{R}^3$ can be modeled by

$$\check{\mathbf{a}}_b = \mathbf{D}^{b/g}\left(\dot{\mathbf{v}}_g^{b/g} - \mathbf{g}_g\right) + \mathbf{w}_b^{ac} \tag{4}$$

where $\mathbf{g}_g \triangleq -g\mathbf{e}_3$ is the gravity acceleration vector, and $\mathbf{w}_b^{ac} \in \mathbb{R}^3$ is assumed to be a zero-mean white Gaussian noise with known covariance $\mathbf{Q}^{ac}$.

Assume that $\dot{\mathbf{v}}_g^{b/g} = 0$ along all the flight, which is a reasonable simplification as one could verify by an MAV flight control simulator [2].

[2] IMAV-M is available in `https://github.com/daviasantos/IMAV-M`.

Magnetometer

Its measure $\check{\mathbf{m}}_b \in \mathbb{R}^3$ can be modeled by

$$\check{\mathbf{m}}_b = \mathbf{D}^{b/g}\mathbf{m}_g + \mathbf{w}_b^{mg} \tag{5}$$

where $\mathbf{m}_g \in \mathbb{R}^3$ is the $\mathcal{S}_g$ representation of the local magnetic field and $\mathbf{w}_b^{mg} \in \mathbb{R}^3$ is assumed to be a zero-mean white Gaussian noise with known covariance $\mathbf{Q}^{mg}$.

Assume that $\mathbf{m}_g$ keeps constant along the drone flight.

Problem statement. . .

Problem Statement

Problem

The problema is to estimate $\alpha^{b/g}$ and β_b^{gy}, by a recursive approximately optimal filter, using:

- the models (1)–(5) and
- the measurements $\check{\mathbf{a}}_b$, $\check{\boldsymbol{\omega}}_b$, and $\check{\mathbf{m}}_b$.

Problem solution. . .

Problem Solution

State Equation

The models (1)–(3) can be put together in the following state equation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) + \mathbf{G}(\mathbf{x})\,\mathbf{w}, \tag{6}$$

where

$$\mathbf{x} \triangleq \left[\left(\boldsymbol{\alpha}^{b/g}\right)^{\mathrm{T}} \; \left(\boldsymbol{\beta}_b^{gy}\right)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^6$$

$$\mathbf{u} \triangleq \check{\boldsymbol{\omega}}_b \in \mathbb{R}^3$$

$$\mathbf{w} \triangleq \left[\left(\mathbf{w}_b^{gy}\right)^{\mathrm{T}} \; \left(\mathbf{w}_b^{d,gy}\right)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^6$$

Problem Solution

State Equation (*cont.*)

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \triangleq \left[\begin{array}{c} \mathbf{A}\left(\boldsymbol{\alpha}^{b/g}\right)\left(\check{\boldsymbol{\omega}}_b^{b/g} - \boldsymbol{\beta}_b^{gy}\right) \\ \mathbf{0}_{3\times 1} \end{array} \right] \in \mathbb{R}^6$$

$$\mathbf{G}(\mathbf{x}) \triangleq \left[\begin{array}{cc} -\mathbf{A}\left(\boldsymbol{\alpha}^{b/g}\right) & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{I}_3 \end{array} \right] \in \mathbb{R}^{6\times 6}$$

Problem Solution

Measurement Equation

Equations (4) and (5) can be put together into the following discrete-time measurement equation:

$$\mathbf{y}_k = \mathbf{h}\left(\mathbf{x}_k\right) + \mathbf{v}_k \tag{7}$$

where (for simplicity, we omit the time t_k from the time-varying quantities)

$$\mathbf{y}_k \triangleq \begin{bmatrix} \check{\mathbf{a}}_b/g \\ \check{\mathbf{m}}_b/\|\mathbf{m}_g\| \end{bmatrix}, \quad \mathbf{h}(\mathbf{x}_k) \triangleq \begin{bmatrix} \mathbf{D}\left(\boldsymbol{\alpha}^{b/g}\right)\mathbf{e}_3 \\ \mathbf{D}\left(\boldsymbol{\alpha}^{b/g}\right)\mathbf{n} \end{bmatrix}, \quad \mathbf{v}_k \triangleq \begin{bmatrix} \mathbf{w}_b^{ac}/g \\ \mathbf{w}_b^{mg}/\|\mathbf{m}_g\| \end{bmatrix}$$

$$\mathbf{D}\left(\boldsymbol{\alpha}^{b/g}\right) \triangleq \mathbf{D}_3(\psi)\mathbf{D}_2(\theta)\mathbf{D}_1(\phi), \quad \mathbf{n} \triangleq \mathbf{m}_g/\|\mathbf{m}_g\|$$

with $\phi \triangleq \mathbf{e}_1^{\mathrm{T}}\boldsymbol{\alpha}^{b/g}$, $\theta \triangleq \mathbf{e}_2^{\mathrm{T}}\boldsymbol{\alpha}^{b/g}$, $\psi \triangleq \mathbf{e}_3^{\mathrm{T}}\boldsymbol{\alpha}^{b/g}$, and $\mathbf{e}_i$, $\forall i$, denoting the standard unit vectors.

References. . .

References

Santos, D. A. **Dynamic Modeling and Control of Multicopters. Chapter 4.** Available in `www.professordavisantos.com`. ITA, 2020.